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On Relativistic Propagators for Large-scale Transport of Cosmic Rays

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ABSTRACT*:* Commonly used cosmic ray transport equations originate in the standard diffusion model as the most known random walk process. However, at least two its features are incompatible with such fixed facts as relativistic boundedness of velocity and multiscale heterogeneity of turbulent magnetic fields. Here is considered a new transport model called the nonlocal relativistic diffusion (NORD) model which is free from both these imperfections. Numerical comparison with some other models confirmed advantage of the NORD model.

KEYWORDS*:* cosmic ray diffusion, nonlocal operators, fractional derivative, relativistic diffusion

I. Introduction

In the frame of large-scale structure of the Universe, cosmic rays (CRs) can be considered as a component of the intergalactic medium, which originate and accelerate inside galaxies, leave them, fill the intergalactic space and have possibility to enter other galaxies and continue acceleration. Multiple repeating, this process may, in principle, produce CRs with ultra-high energy. To estimate characteristics of this process, we follow conventional way, separating the process into two aspects: CR penetration with a fixed energy serving as a parameter, and CR energy distribution evolution due to acceleration in turbulent magnetic fields. This article considers only the first aspect, solving of which needs some revision of existing propagators in order to choose or modify some of them adapting it to the intergalactic transport problem. As is known, the diffusion approximation fails to describe such characteristics of the process as time- and path-distributions, playing a more important róle in the intergalactic transport. At least two features distinguish this process from conventional models for interstellar transport: the presence of long free path sections on CR trajectories and relativistic boundedness of CR velocity (recall that the latter is infinite in the Brownian diffusion model) [1, 2]. In this article, we discuss some new propagators being free from such imperfections including original one. The latter is derived by combining two above-mentioned peculiarities leading to the fractional-order differential equation with the material derivative operator. Due to the fractional-order operator belongs to the nonlocal family, we call this process the *nonlocal relativistic diffusion* (NORD).

II. Propagators: from ordinary to relativistic nonlocal diffusion

Let us list the propagator models, which we compare in estimations of equilibrium spectra in this work

- ***The Gaussian density function***

$$G_{\mathrm{G}}(\mathbf{r},t,E;\mathbf{r}_0,t_0,E_0)=\frac{\delta(E-E_0)}{[4\pi D(E)\tau]^{3/2}}\exp\left[-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{4D(E)\tau}\right];\quad \tau=t-t_0, \qquad (1)$$

is the fundamental solution of the ordinary diffusion equation. The model doesn't take into account neither the relativistic boundedness of speed nor fractal properties of ISM. First, it was applied to CR transport in 60th years of the former century.

- ***The Dunkel-Hänggi-Jüttner (DHJ) propagator*** [3-5]

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$$G_{\mathrm{J}}(\mathbf{r},t,E;\mathbf{r}_0,t_0,E_0)=\frac{N_k(E)1(c\tau-r)c^2\tau}{8D(E)\pi(c\tau)^3K_1\left(\frac{c^2\tau}{2D(E)}\right)\left[1-\frac{r^2}{(c\tau)^2}\right]^2}\exp\left(-\frac{c^2\tau}{2D(E)\sqrt{1-\frac{r^2}{(c\tau)^2}}}\right),\quad (2)$$

$$\tau=t-t_0,\ r=|\mathbf{r}-\mathbf{r}_0|,$$

includes the relativistic boundedness and describes both regimes: small-scale diffusive motion and ballistic motion as particular extreme cases. Its first application to CR-transport was undertaken in [1]. However, this propagator does not account for multiscale heterogeneity (fractality) of ISM.

- ***The Lévy-Feldheim trivariate isotropic distributions*** $g_3(r,\alpha)$, $\alpha\in(0,2]$:

$$G_{\mathrm{LF}}(\mathbf{r},t,E;\mathbf{r}_0,t_0,E_0)=\delta(E-E_0)[K(E)(t-t_0)]^{-3/\alpha}g_3\left(|\mathbf{r}-\mathbf{r}_0|[K(E)(t-t_0)]^{-1/\alpha}\right),t>t_0,$$

are the Green functions of the diffusion equation with the fractional Laplacian, describing superdiffusive transport of CRs [6]. It is important to note, that when passing from the normal value $\alpha=2$ to anomalous ones $\alpha<2$ it is not only the form what is changed (power tails appears and increases) but the extension rate of diffusion packet also changes (increases): its width grows with time $\tau=|t-t_0|$ accordingly to the law $\tau^{1/\alpha}$, $\alpha<2$, while in the normal case the packet's width is proportional to $\tau^{1/2}$. These propagators introduced into CR-transport in the Galaxy in [6, 7] to solve the energy spectrum problem, taking into account the ISM heterogeneities were testified to take large-scaled (fractal) character (see [8]). The supernova remnants analysis shows the presence in this region of gas components with different physical parameters (T_e : $5\div10^6$ K, n_e : $0.1\div10^3$ m $^{-3}$), what can be the sequence of turbulent heterogeneity of ISM. These facts and other data concerning the heterogeneities of matter density ρ and magnetic field intensity $H\propto\rho^q$, q : $1/3\div1/2$ ([9]) within the length range 100-150 pc, giving rise to uncertainties of diffusion model to be applicable to cosmic rays transfer description, stimulated the use of the superdiffusion model, based on the fractional Laplacian operator [10]. However, this model assumes instantaneous jumps interrupted by confining in magnetic traps.

In a single-nearest source approximation, this model turned out to be able to explain the experimentally observed 'knee' in the energy spectrum, i. e. increasing of the exponent η in power representation of the $E^{-\eta}$ spectrum while passing from the region : 10^2 Gev/nucleon into that of : 10^5 Gev/nucleon. The explanation is connected with the presence of the anomalously large free paths of particles (Lévy "flights") in the regions of reduced magnetic fields.

- ***Nonlocal relativistic diffusion (NORD) propagator*** accounting for both these effects (finite velocity and fractality of ISM) was proposed in [2] and developed in our papers (see, e.g. [11, 12]). It is based on integral equations of the form

$$G_{\mathrm{NR}}(\mathbf{r},t,E)=\int\frac{d\mathbf{r}'}{v}P(\mathbf{r}',E)f\left(\mathbf{r}-\mathbf{r}',t-\frac{r'}{v},E\right),\quad (3)$$

$$f(\mathbf{r},t,E)=\int f\left(\mathbf{r}-\mathbf{r}',t-\frac{r'}{v},E\right)p(\mathbf{r}',E)d\mathbf{r}'+\delta(\mathbf{r}')\delta(t),\quad (4)$$

where $f(\mathbf{r},t,E)$ is the density of collisions per a unit volume and a unit time, $p(\mathbf{r}',E)$ and $P(\mathbf{r}',E)$ are density function and survival probability for free path lengths. Under appropriate choice of transtion probability P, this model, including an integral (i.e. nonlocal) operator is able to describe relativistic kinetics in a turbulent media.

To model multiscale heterogeneity, we use power law distribution of free path length

$$p(r,E) = \alpha r_0^\alpha (E) r^{-\alpha-1}, \quad \alpha > 1.$$

The corresponding asymptotical part of these integral equations was found in [2] for 3D case in the form

$$\frac{R_{1/2}^\alpha}{2}\left\langle\left(v^{-1}\frac{\partial}{\partial t}+\vec{\Omega}\nabla\right)^\alpha\right\rangle n(\mathbf{r},t) = v^{-1}S(\mathbf{r},t), \quad 0<\alpha<1. \tag{5}$$

$$\langle R\rangle\left\langle\left(v^{-1}\frac{\partial}{\partial t}+\vec{\Omega}\nabla\right)\right\rangle n(\mathbf{r},t) - \frac{R_{1/2}^\alpha}{2}\left\langle\left(v^{-1}\frac{\partial}{\partial t}+\vec{\Omega}\nabla\right)^\alpha\right\rangle n(\mathbf{r},t) = v^{-1}S(\mathbf{r},t), \quad 1<\alpha<2. \tag{6}$$

where $\langle R\rangle$ is the mean path length, $R_{1/2}$ the median, $S(\mathbf{r},t)$ the source term (see details in [2]). These nonlocal diffusion equations contain fractional material derivative of order α averaged over propagation directions. The 1D, 2D and 3D versions are included in this system as particular cases [2].

III. SOME RESULTS OF NUMERICAL CALCULATIONS

Here, we evaluative the background spectrum expected from CR sources distributed in the Galactic disc using propagators listed in the previous section. Source spectrum is usually modeled by a power law with exponential cutoff [13]. We consider a pure power law for injected particles.

Diffusion coefficient, "determined from statistically reliable data (up to 100 GeV/nucleon) on secondary nuclei" [14], depends on energy according to the power law $\propto R^\delta$, where $R = pc/Z$ is a magnetic rigidity, p momentum of a particle, Z nucleus charge, $\beta = v/c$, v particle velocity. Exponent $\delta = 0.6$ in pure diffusion model, and $\delta = 0.34$ accounting for diffusive reacceleration of particles by random magnetohydrodynamic waves in interstellar medium and Alfven velocity $V_a \approx 36 \text{ km}\cdot\text{s}^{-1}$ [15]. Popular dependencies for diffusion coefficients in different models for $R > R_0 = 3$ GV are listed in Table 1.

Table 1: Diffusion coefficients in different models for $R > R_0 = 3$ GV.

Model	Diffusion Coefficient D [cm^2/s]	$\breve{\lambda}_{10^6\text{GV}}$ [pc]
Ptuskin et al. (plain diffusion) [15]	$2.2\cdot10^{28}\beta^{-2}(R/R_0)^{0.6}$	1468
Ptuskin et al. (diffusive reacceleration) [15]	$5.2\cdot10^{28}\beta^{-2}(R/R_0)^{0.34}$	127
Ptuskin et al. (diffusive reacceleration with damping) [15]	$2.9\cdot10^{28}\beta^{-2}(R/R_0)^{0.5}$	542
Blasi and Amato, $\delta = 0.6$ [13]	$0.55\cdot10^{28}H_{\text{kpc}}(R/R_0)^{0.6}$	1101
Blasi and Amato, $\delta = 1/3$ [13]	$1.33\cdot10^{28}H_{\text{kpc}}(R/R_0)^{1/3}$	90

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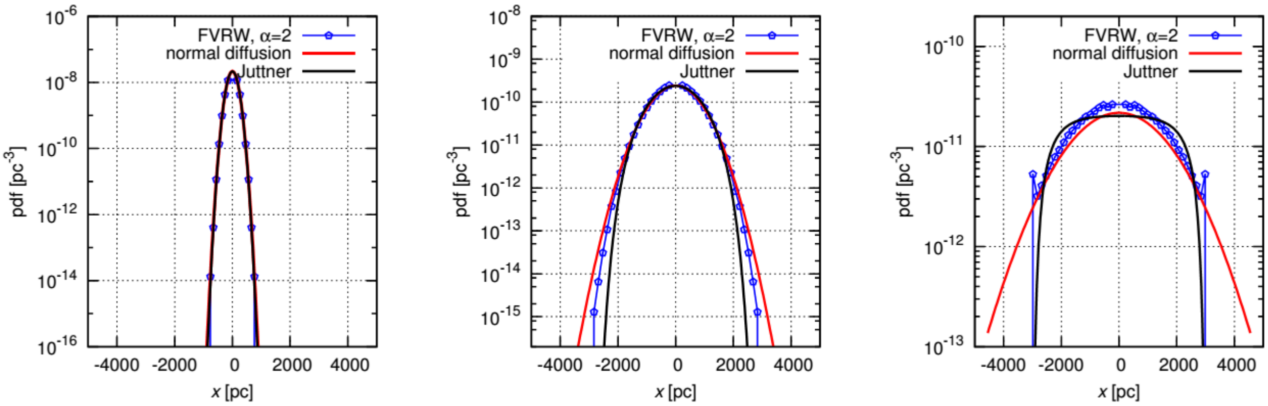

Fig. 1: Comparison of evolutions of the DHJ distribution with the Monte Carlo solutions of the constant-velocity random walk for the case of exponential distribution of path lengths. Results for three values of rigidity are presented $R = 1 \cdot 10^4$ GV (left panel), $R = 5 \cdot 10^4$ GV (middle panel) $R = 3 \cdot 10^5$ GV (right panel); $\delta = 0.6$ (plain diffusion). The age is 50000 yr.

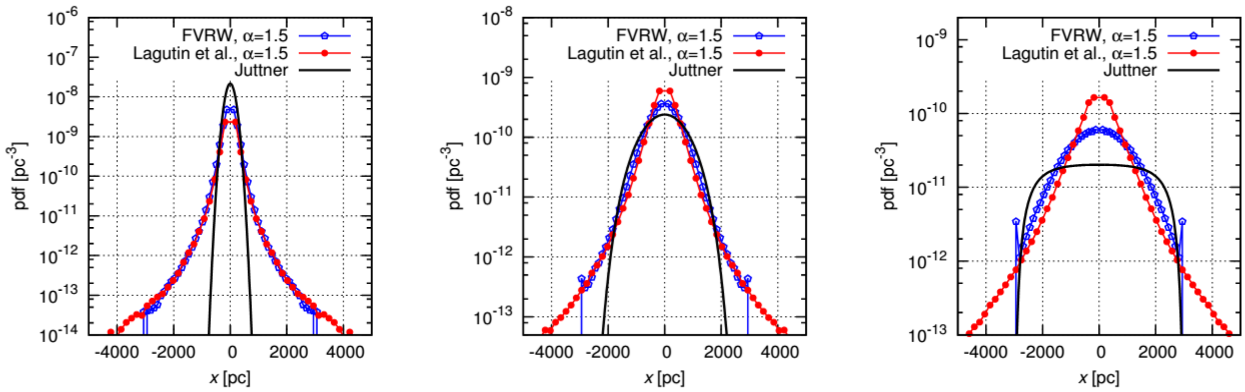

Fig. 2: Comparison of evolutions of the DHJ distribution with the Monte Carlo solutions of the constant-velocity random walk for the case of power law distribution ($\alpha = 1.5$) of path lengths. Results for three values of rigidity are presented $R = 1 \cdot 10^4$ GV (left panel), $R = 5 \cdot 10^4$ GV (middle panel) $R = 3 \cdot 10^5$ GV (right panel); $\delta = 0.6$ (plain anomalous diffusion). The age is 50000 yr.

Relativistic restriction plays a crucial role when the size of propagation area of interest is comparable with a mean free path length λ. The latter may be estimated as a function of energy $\lambda(E)$ from the acceptable values of the diffusion coefficient used, for example, in GALPROP [15] or in some other calculations, e.g. presented in [13, 14]. The lower-bound estimate for isotropic three-dimensional diffusion can be found via formula $\breve{\lambda}(E) = 3D(E)/c$, where c is the speed of light. Calculations show that the mean free path at rigidity 10^6 GV may be comparable with the size of the Galactic halo ($H = 4$ kpc) in some of the mentioned models (see table). This indicate that relativistic boundedness of propagators may play role in formation of the spectrum near the 'knee'. Comparison of the listed propagators is presented in Figs. 1 and 2.

Evaluative calculations of the spectrum expected from CR sources distributed in the Galactic disc is performed in the following way. The disc is considered as infinitely thin and the sources are uniformly distributed in this disk with radius R_d. The flux of CRs propagating in the space surrounding the disk is calculated at the center according to formula

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$$n_{\mathrm{CR}} = \int_0^{R_{\mathrm{d}}} dr \int_{t_{\mathrm{in}}}^{\infty} d\tau \, \frac{2\pi\, r}{\pi R_{\mathrm{d}}^2} N(E)\, \mathrm{P} \;\; G(\mathbf{r},t,E), \qquad (7)$$

where $t_{\mathrm{in}} = 0$ is taken by Blasi and Amato [13]. Here, P is a rate of SN explosions. We estimate the spectrum of protons (no spallation). We do not account for the Galactic halo boundaries in these evaluating calculations. The Gaussian and Lévy-Feldheim propagators admit nonzero values beyond the relativistic restriction. In these cases, the lower limit $t_{\mathrm{in}} = 0$ can lead to incorrect results. Limit $t_{\mathrm{in}} = r/v$ implying the ballistic restriction of propagators is more correct.

Here, we imply the standard assumptions about the origin of galactic cosmic rays [16]. First, CRs are injected from Galactic sources, such as supernova remnants (SNRs) with power law spectra, $N(E) \sim E^{-\gamma}$. The second point, cosmic rays propagate diffusively throughout the Galaxy with a diffusion coefficient $D(E) \propto E^{\delta}$. The simple combination of these propositions leads to the equilibrium spectrum $n(E) \propto N(E)/D(E) \propto E^{-\gamma+\delta}$ reflecting the balance between injection and escape of CRs from the confinement volume (e.g. the galactic halo) [16]. Such estimation is modified in the NORD model.

Results of calculations for proton spectra for various propagators are presented in Fig. 3. We considered Gaussian, DHJ, renormalized and truncated Gaussian and Lévy-Feldheim propagators approximating solutions of nonlocal relativistic diffusion equation. Remarkable result is that accounting for relativistic restriction steepens the spectrum at high energies. Under assumption that CRs are injected from sources with pure power law spectra, $N(E) \sim E^{-\gamma}$, the position of the spectrum break, its sharpness and slopes of asymptotics depend on diffusion parameters $D_{\alpha}(E)$ and α (see Fig. 4, right panel).

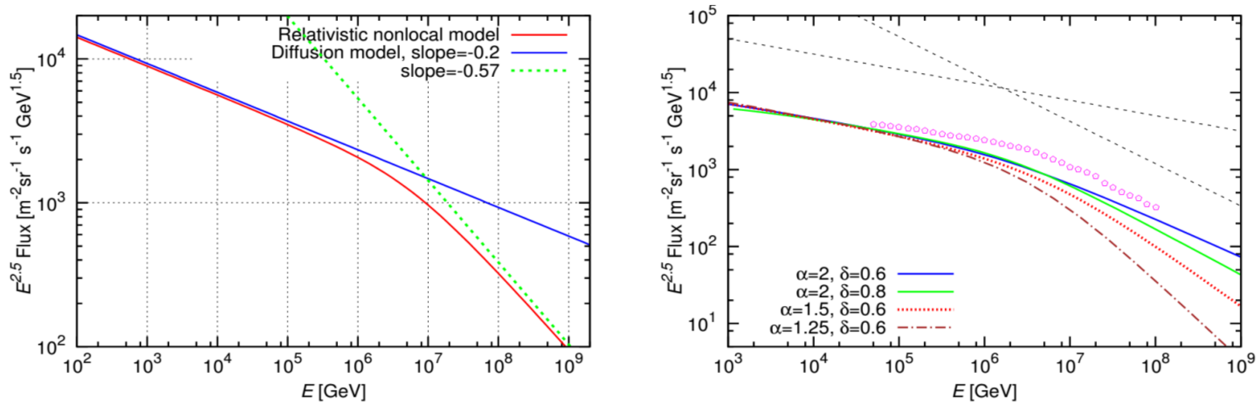

Fig. 3: Estimates of equilibrium spectra. Left panel: spectra calculated for the case of $D(E) \propto E^{\delta}$ with $\delta = 0.75$, and $N(E) \propto E^{-\gamma}$ with $\gamma = 1.95$ for the Gaussian nontruncated propagator (blue line) and the relativistic one (red line). Right panel: spectra for truncated Gaussian and Lévy-Feldheim propagators ($\alpha = 1.5,\ 1.25$) approximating solutions of nonlocal relativistic diffusion equation ($v = c/3$).

So, the NORD model provides an alternative cause of the spectrum steepening at high energies. Sure, the determination of position of the break provided by relativistic effect, its role in formation of the knee observed in experiment requires additional detailed calculations and matching. But obvious, that cause of this effect is universal and must be kept in mind in all calculations of propagation of high energy CRs. Explanation of the spectrum steepening at

high energies in the NORD model is schematically represented in Fig. 4. The grey field corresponds to the space-time area containing CR sources which do not take part in the formation of the spectrum due to the relativistic restriction. Their accounting in the ordinary diffusion model leads to a pure power law spectrum $\propto E^{-\gamma-\delta}$ under the

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assumption about the pure power law spectrum of injected particles. Taking the relativistic restriction into account in the NORD model excludes these sources from the consideration, but this exclusion affects predominantly on high energy part of CRs due to $D(E) \propto E^{\delta}$. Number of these sources is not small. The lower-bound estimate for $R_d = 15$ kpc, $\mathrm{P} = 1/100$ yr $^{-1}$, is about 300 young sources (with ages □45000 yr). Mentioned truncation procedure also implies that we neglects by sources located at the boundary of the relativistic cone, i.e. there are no CR fronts from quite young sources penetrating the Sun system at the moment.

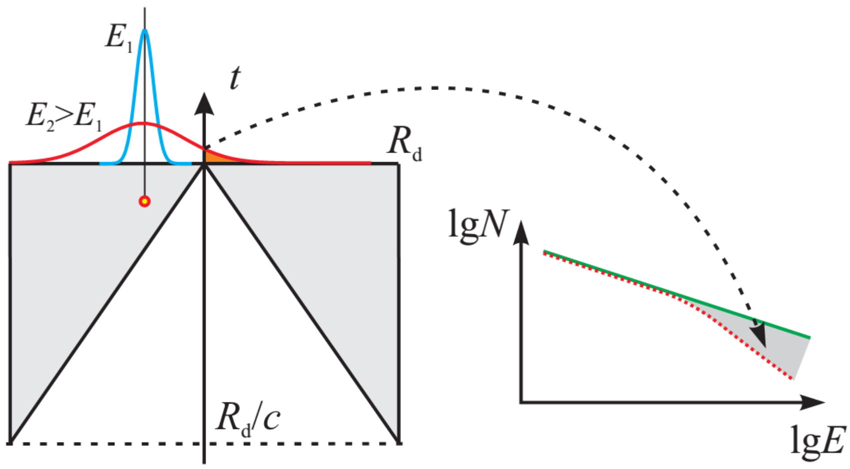

Fig. 4: Sketch providing a possible explanation of steepening in the spectrum at high energies

IV. CONCLUSION

The NORD model discussed in this article accounts for the relativistic boundedness of CR particle velocity, nonlocal character of particle transport caused by the turbulent nature of the space magnetic field, energy dependence of the mean free path length. First calculations in frame of this model demonstrates the following.

1. The DHJ propagator is based on the Gibbs statistics derived for Gaussian random fields with short correlations. For this reason it differs from ordinary diffusion model only at short times.

2. Contribution of nearest sources into the 'knee' formation doesn't practically depend on the relativistic boundedness of speed but essentially determined by fractal property of ISM, as it was obtained in [2].

3. On the contrary, the averaged influence of distant sources depends rather on the speed boundedness than on turbulent character of ISM. Moreover, this fact brings some contribution into the 'knee' formation as well. The remarkable result is that under conventional assumptions about the origin of galactic cosmic rays the relativistic principle of speed boundedness may explain the origin of 'knee' in the equilibrium spectrum of cosmic rays. Theoretical 'knee' energy depends on a propagator model and corresponding energy dependence of the diffusion coefficient. Calculations show that the ordinary diffusion model with commonly used parameters leads to E_{knee} values (produced by relativistic restriction) a little greater than the observed ones.

These results allow us to get to the conclusion that the NORD propagator may serve as an appropriate tool for solving large-scale (both interstellar and intergalactic) transport problems.

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