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Polynomially Paranormal Operators on the Tensor Product of Hilbert Spaces

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ABSTRACT: In this paper we have discuss about the hyponormal and paranormal operators in the Hilbert Space and their tensor product. Here an operator is constructed by taking the composition of two paranormal operators on the tensor product of Hilbert spaces and derived the conditions for polynomially paranormal operators.

KEYWORDS:Paranormal Operators, Polynomially Paranormal Operator,Algebraic Tensor Product, Projective Tensor Norm.

I. INTRODUCTION

A paranormal operator is a generalization of a normal operator.The class of paranormal operators was introduced by V. Istratescu in 1960s, though the term "paranormal" is probably due to Furuta(1967). Every hyponormal operator (in particular a quasi-normal operator and a normal operator) is paranormal. Here we examine some properties of polynomially paranormal operators on the Hilbert Space and their Tensor product.Throughout this paper,$\beta(H)$ denotes the set of all bounded linearoperators acting on a complex Hilbert space$H$. In this section some definitions and preliminary results are discussed.

Definition 1.1: An operator $T \in \beta(H)$ is called an n-normal operator if $T^nT^* = T^*T^n$.
Lemma 1.2. [Fuglede Theorem] If T is an operator then $T^nT = T\,T^n$ for any $n \in \mathbb{N}$.
Lemma 1.3. Let $T \in \beta(H)$.Then $T$is n-normal if and only if$T^n$is normal.
Proof. Let $T$is n-normal, $T^nT^* = T^*T^n$. Therefore
$T^n(T^*)^n = T^*T^n(T^*)^{n-1} = T^*(T^nT^*)(T^*)^{n-2} = (T^*)^2T^n(T^*)^{n-2} = (T^*)^nT^n$.
Then $T^n$is normal. Now, let $T^n$is normal.
Since $T^nT = TT^n$, by Fuglede theorem , $T^nT^* = T^*T^n$.
Therefore T is n-normal.

Now, it is clear that
normal $\Rightarrow$ n-normal for any n.
But the converse is not always true.
If $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$, then T is 2-normal but not normal.
All nonzero nilpotent operators are n-normal operators, for $n \geq k$where $k$the index of nilpotance, but they are not normal.
Lemma 1.4: $T$is n-normal if and only if$\| T^nx \| = \| (T^n)^*x \| \quad \forall x \in H$.

Definition 1.5. An operator T on a Hilbert Space H is said to be hyponormal if $T^*T \geq T\,T^*$which is equivalent to $\| T^*(x) \| \leq \| T(x) \|$, $\forall x \in H$.
An operator $T$is quasi hyponormal if $(T^*)^2T^2 \geq (T^*T)^2$ holds, which is equivalent to $\| T^*T(x) \| \leq \| T^2(x) \|$, $\forall x \in H$.
An operator $T$is said to be M- hyponormal if there exists a positive real number M such that

$$M^2(T-\lambda)^*(T-\lambda) \geq (T-\lambda)(T-\lambda)^* \text{for all } \lambda \in \mathbb{C}.$$

An operator T on a Hilbert Space $H$is said to be paranormal if $\| T(x) \|^2 \leq \| T^2(x) \|$, for every unit vector $x$in H.
Another equivalent definition of paranormal operator in terms of topological aspects can be given as:
An operator Ton a Hilbert Space $H$is said to be paranormal if $\| (T-zI)^{-1} \| = 1/d(z,\sigma(T))$ for all $z \notin \sigma(T)$ where $d(z,\sigma(T))$ is the distance from z to $\sigma(T)$,the spectrum of T.
An operator T on a Hilbert Space $H$is said to be M $-$paranormal if, $T(x) \|^2 \leq M \| T^2(x) \|$,for every unit vector $x$in $H$and for some positive real number M.

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An operator T on a Hilbert Space H is said to be n-paranormal (for some positive integer n) if
$\| T(x) \|^n \leq \| T^n(x) \| \| x \|^{n-1}, \forall x \in H$.
An operator T is called polynomially paranormal if there exists a non-constant polynomial $q(z)$ such that $q(T)$ is paranormal.
In general the following implication holds:
normal $\Rightarrow$ n-normal $\Rightarrow$ hyponormal $\Rightarrow$ paranormal $\Rightarrow$ n-paranormal.
But the converse is not true.

Definition 1.6. For $T \in \beta(H)$, $R(T)$ and $N(T)$ denotes the range and the null space of T, respectively.
An operator $T \in \beta(H)$ is said to have finite ascent if $N(T^m) = N(T^{m+1})$ for some positive integer m, and finite descent if $R(T^m)) = R(T^{m+1})$ for some positive integer m.

Definition 1.7. Let X, Y be normed spaces over $\mathbb{K}$ with dual spaces X^*, Y^*. Given $x \in X, y \in Y$, let $x \otimes y$ be the element of $BL(X^*, Y^*; \mathbb{K})$, the set of all bounded bilinear functionals from $X^* \times Y^*$ to $\mathbb{K}$, and is defined by

$$(x \otimes y)(f, g) = f(x)g(y) \ (f \in X^*, g \in Y^*)$$

The algebraic tensor product of X and Y, $X \otimes Y$ is defined to be linear span of
$\{x \otimes y : x \in X, y \in Y\} \ in \ BL(X^*, Y^*; \mathbb{K})$. (refer to [4])

Definition 1.8. Given normed spaces X and Y, the projective tensor norm γ on $X \otimes Y$ is defined by

$$\gamma(u) = \inf\left\{\sum_i \| x_i \| \| y_i \| : u = \sum_i x_i \otimes y_i\right\}$$

where the infimum is taken over all finite representations of u.
The completion of $X \otimes Y$ with respect to γ is called the projective tensor product of X and Y and is denoted by $X \otimes_\gamma Y$.
If X and Y are Hilbert spaces, an inner product on $X \otimes_\gamma Y$ is defined as

$$\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$$

where $a, c \in X$ and $b, d \in Y$.
Then it can be shown that $X \otimes_\gamma Y$ is a Hilbert space.

Lemma 1.9: Given $u \in X \otimes Y$, there exists linearly independent sets $\{x_i\}, \{y_i\}$ such that $u = \sum_n x_i \otimes y_i$.
Lemma 1.10: $X \otimes_\gamma Y$ can be represented by as the linear subspace of $BL(X'; Y'; \mathbb{K})$ consisting of all elements of the form $u = \sum_n x_n \otimes y_n$ where $\sum_n \| x_n \| \| y_n \| < \infty$.
Moreover $\gamma(u)$ is the infimum of the sums $\sum_n \| x_n \| \| y_n \| < \infty$ over all such representation of u.
Lemma 1.11: If T is a paranormal, then T^n is paranormal.
Lemma 1.12: A compact paranormal operator is normal.
Lemma 1.13: There exists an invertible paranormal operator T such that
1. T is not hyponormal.
2. T^2 is not paranormal.
3. $\| T \| \geq R_{sp}(T)$; $R_{sp}(T)$ denotes spectral radius of .
4. T^{-1} is not paranormal.

Lemma 1.14: If $T \in B(H)$ is polynomially paranormal, then $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.

Lemma 1.15: *If S and T are paranormal operators then $S + T$ is paranormal if it satisfies*

$$S^*AS \geq 0 \quad T^*BT \geq 0 \quad S^*AT \geq 0 \quad S^*BT \geq 0 \quad T^*AS \geq 0 \quad T^*BS \geq 0$$

where $A = T^*T - TT^*$ & $B = S^*S - SS^*$

Lemma 1.16: *If S and T are two commuting paranormal operators then TS* is paranormal under the condition,

$$\max\{\| S(x) \|^2, \| T(x) \|^2\} \leq \| x \|$$

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II. RESULTS

We construct an operator T on the tensor product of Hilbert spaces.Let T_1 be an operator on X and T_2 be an operator on Y.

We define $T: X \otimes_\gamma Y \longrightarrow X \otimes_\gamma Y$ by

$$T\left(\sum_i x_i \otimes y_i\right) = \sum_i T_1(x_i) \otimes T_2(y_i)$$

Now an immediate result follows considering T_1 and T_2 to be paranormal.

Theorem 2.1: If T_1 and T_2 are paranormal then T is paranormal.

Proof:

$$\| T(\Sigma_i x_i \otimes y_i) \|^2 = \| \Sigma_i T_1(x_i) \otimes T_2(y_i) \|^2$$
$$\leq \Sigma_i \| T_1(x_i)) \|^2 . \| T_2(y_i) \|^2$$

$\leq \Sigma_i \| T_1^2(x_i) \| . \| {T_2}^2(y_i) \|$ [$\because$ T_1 and T_2 are paranormal]

Thus, $\| T(\sum_i x_i \otimes y_i) \|^2 \leq \Sigma_i \| T_1^2(x_i) \otimes T_2^2(y_i) \|$ [Taking projective norm]

$$= \| T^2(\Sigma_i x_i \otimes y_i) \|$$

So, T is paranormal.□

Lemma 2.2: Let $T \in \beta(H)$ be polynomially paranormal. Then $N(T - \lambda) = N((T - \lambda)^2)$ for $\lambda \in \mathbb{C}$.

Proof. Let $x \in N((T - \lambda)^2)$. Since $N(T - \lambda) \subset N((T - \lambda)^*)$ for $\lambda \in \mathbb{C}$..

We have $(T - \lambda)x \in N(T - \lambda) \subset N((T - \lambda)^*)$.

Hence

$$\| (T - \lambda)x \|^2 = \langle (T - \lambda)^*(T - \lambda)x; x \rangle = 0$$

Hence $N((T - \lambda)^2) \subset N(T - \lambda)$.

The converse follows. □

Lemma 2.3: *If* $T \in \beta(H)$ is polynomially paranormal, then $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.

An operator $T \in \beta(H)$ is said to have the single valued extension property if there exists no nonzero analytic function f such that $(T - z) f(z) \equiv 0$. Larusen [14] proved that if $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, then T has the single valued extension property.

Theorem 2.4. If T_1 and T_2 is polynomially paranormal and T_2 be a projection of Y then $T - \lambda$ has finite ascent on $X \otimes Y$ for all $\lambda \in \mathbb{C}$.

Proof. Since T_1 is polynomially paranormal so by lemma [2.2], $T_1 - \lambda$ has finite ascent on X for all $\lambda \in \mathbb{C}$.

$\therefore N((T_1 - \lambda)^p) = N((T_1 - \lambda)^{p+1})$, for some positive integer p,

And $N((T_2 - \lambda)^p) = N((T_2 - \lambda)^{p+1})$, for some positive integer p.

Let $u = \Sigma_i x_i \otimes y_i \in X \otimes Y$ be such that $u \in N((T - \lambda)^p)$.

$$\therefore (T - \lambda)^p(u) = 0 \Rightarrow (T - \lambda)^p(\Sigma_i x_i \otimes y_i) = 0$$
$$\Rightarrow \Sigma_i((T_1 - \lambda)^p x_i \otimes (T_2 - \lambda)^p y_i = 0 \quad [\because y_i \in Y, \text{so}, (T_2 - \lambda)(y_i) = y_i]$$
$$\Rightarrow x_i \in N((T_1 - \lambda)^p) \text{or } y_i = 0 \ \forall i$$
$$\Rightarrow x_i \in N((T_1 - \lambda)^{p+1}) \text{or } y_i = 0 \ \forall i$$
$$\Rightarrow (T_1 - \lambda)^{p+1} x_i \otimes y_i = 0 \ \forall i$$
$$\Rightarrow \Sigma_i (T_1 - \lambda)^{p+1} x_i \otimes y_i = 0$$
$$\Rightarrow (T - \lambda)^{p+1} \Sigma_i x_i \otimes y_i = 0$$
$$\Rightarrow \Sigma_i x_i \otimes y_i \in N((T - \lambda)^{p+1})$$

Thus, $N((T - \lambda)^p) \subseteq N((T - \lambda)^{p+1})$.

Similarly we can show $N((T - \lambda)^{p+1}) \subseteq N((T - \lambda)^p)$.

Therefore $T - \lambda$ has finite ascent. □

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III.CONCLUSION

Here we have discussed on polynomially paranormal operators, now the question can we derive results for the *-n-paranormal operators which was defined by Z. Lingling & A. Uchiyama.
Let X be a complex Banach space. Let $\pi(x)$ be

$$\pi(x) = \{(x, f) \in X \times X^*: \| f \| = f(x) = \| x \| = 1\}$$

Where X^* is the dual of X.
$T \in B(X)$ is said to be ***-n-**paranormalif $\| T^* f \|^n \leq \| T^n x \|$for all $(x, f) \in \pi(X)$,where $T^*$is the dual operator of T. Moreover it is to be derived that whether the result for single valued extension property holds for *-n-paranormal operators.

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