ABSTRACT: The notion of quasiinjectivity relative to a class of finitely generated subsystems namely finitely quasi injective and quasi finitely injective systems over monoids are introduced and studied which are proper generalizations of quasi injective systems . Several properties of these kind of generalizations are discussed . Conditions under which subsystems of finitely quasi injective system inherit this property . Characterizations of finitely injective and quasi finitely injective systems over monoids are considered . The relationship between the classes of finitely quasi injective with other classes of injectivity are studied . As a consequence, conditions to versus these classes are shown .

KEYWORDS: Finitely quasi-injective systems , Quasi finitely injective systems, Finitely injective systems , Quasi finitely injective systems over monoids .

I-INTRODUCTION AND PRELIMINARIES

Throughout this paper , the basic S-system is a unitary right S-system with zero which is consists of a monoid with zero , a non-empty set M , with a function f : M × S → M such that f(m,s) = ms and the following properties hold : (1) m · 1 = m (2) m(st) = (ms)t , where 1 is the identity element of S . An element Θ ∈ M is called fixed of M if Θ = Θ for all s ∈ S [4] . An S-system MΘ is centered if it has a fixed element Θ necessary unique such that mΘ = Θ for all m ∈ MΘ , where 0 is the zero element of S and Θ is the zero of M [8] . A subsystem N of an S-system MΘ , is a non-empty subset of M such that x ∈ N for all x ∈ N and s ∈ S [8] . Let g be a function from an S-system A into an S-system B , then g will be called an S-homomorphism , if for any a ∈ A and s ∈ S , we have g(as) = g(a)s [3] . An S-congruence ρ on a right S-system MΘ is an equivalence relation on MΘ such that whenever (a,b) ∈ ρ , then (as, bs) ∈ ρ for all s ∈ S [6] . The identity S-congruence on MΘ will be denoted by IMΘ such that (a,b) ∈ IMΘ if and only if a = b [6] .

The authors defined that if for every x ∈ MΘ , there is an S-homomorphism f : MΘ → xS such that x = f(x) for x ∈ MΘ , then an S-system MΘ is called principal self-generator [1] . A subset A of an S-system MΘ is called a set of generating elements or a generating set of MΘ , if every element m ∈ MΘ can be presented as m = as for some a ∈ A , s ∈ S . Then , an S-system MΘ is finitely generated if MΘ = < A > for some A , |A| < ∞ , where < A > is the subsystem of MΘ generated by A [7, p.63] . An S-system NΘ is called MΘ-generated , where MΘ be an S-system if there exists an S-epimorphism α : MΘ → NΘ for some index set I . If I is infinite , then NΘ is called finitely MΘ-generated of MΘ [2] . An S-system BΘ is a retract of an S-system AΘ , if and only if there exists a subsystem W of AΘ and epimorphism f : AΘ → W such that BΘ = W and f(w) = w for every w ∈ W [7, p.84] . An S-homomorphism f which maps an S-system MΘ into an S-system NΘ is said to be split if there exists S-homomorphism g which maps NΘ into MΘ such that fg = 1MΘ [6] .

Let AΘ , MΘ be two S-systems . AΘ is called MΘ-injective if given an S- monomorphism α : NΘ → MΘ where NΘ is a subsystem of MΘ and every S-homomorphism β : NΘ → AΘ , can be extended to an S-homomorphism σ : MΘ → AΘ [10] . An S-system AΘ is injective if and only if it is MΘ-injective for all S-systems MΘ . An S-system AΘ is quasi injective if and only if it is AΘ-injective . Quasi injective S-systems have been studied by Lopez and Luedeman [8] . It is clear that every injective system is quasi injective but the converse is not true in general see [8] . An S-system AΘ is weakly injective if it is injective relative to all embeddings of right ideals into SΘ , [7,p.205] .

ABSTRACT: The notion of quasiinjectivity relative to a class of finitely generated subsystems namely finitely quasi injective and quasi finitely injective systems over monoids are introduced and studied which are proper generalizations of quasi injective systems . Several properties of these kind of generalizations are discussed . Conditions under which subsystems of finitely quasi injective system inherit this property . Characterizations of finitely quasi injective and quasi finitely injective systems over monoids are considered . The relationship between the classes of finitely quasi injective with other classes of injectivity are studied . As a consequence, conditions to versus these classes are shown .

KEYWORDS: Finitely quasi-injective systems , Quasi finitely injective systems, Finitely injective systems , Quasi finitely injective systems over monoids .

I-INTRODUCTION AND PRELIMINARIES

Throughout this paper , the basic S-system is a unitary right S-system with zero which is consists of a monoid with zero , a non-empty set M , with a function f : M × S → M such that f(m,s) = ms and the following properties hold : (1) m · 1 = m (2) m(st) = (ms)t , where 1 is the identity element of S . An element Θ ∈ M is called fixed of M if Θ = Θ for all s ∈ S [4] . An S-system MΘ is centered if it has a fixed element Θ necessary unique such that mΘ = Θ for all m ∈ MΘ , where 0 is the zero element of S and Θ is the zero of M [8] . A subsystem N of an S-system MΘ , is a non-empty subset of M such that x ∈ N for all x ∈ N and s ∈ S [8] . Let g be a function from an S-system A into an S-system B , then g will be called an S-homomorphism , if for any a ∈ A and s ∈ S , we have g(as) = g(a)s [3] . An S-congruence ρ on a right S-system MΘ is an equivalence relation on MΘ such that whenever (a,b) ∈ ρ , then (as, bs) ∈ ρ for all s ∈ S [6] . The identity S-congruence on MΘ will be denoted by IMΘ such that (a,b) ∈ IMΘ if and only if a = b [6] .

The authors defined that if for every x ∈ MΘ , there is an S-homomorphism f : MΘ → xS such that x = f(x) for x ∈ MΘ , then an S-system MΘ is called principal self-generator [1] . A subset A of an S-system MΘ is called a set of generating elements or a generating set of MΘ , if every element m ∈ MΘ can be presented as m = as for some a ∈ A , s ∈ S . Then , an S-system MΘ is finitely generated if MΘ = < A > for some A , |A| < ∞ , where < A > is the subsystem of MΘ generated by A [7, p.63] . An S-system NΘ is called MΘ-generated , where MΘ be an S-system if there exists an S-epimorphism α : MΘ → NΘ for some index set I . If I is infinite , then NΘ is called finitely MΘ-generated of MΘ [2] . An S-system BΘ is a retract of an S-system AΘ , if and only if there exists a subsystem W of AΘ and epimorphism f : AΘ → W such that BΘ = W and f(w) = w for every w ∈ W [7, p.84] . An S-homomorphism f which maps an S-system MΘ into an S-system NΘ is said to be split if there exists S-homomorphism g which maps NΘ into MΘ such that fg = 1MΘ [6] .

Let AΘ , MΘ be two S-systems . AΘ is called MΘ-injective if given an S- monomorphism α : NΘ → MΘ where NΘ is a subsystem of MΘ and every S-homomorphism β : NΘ → AΘ , can be extended to an S-homomorphism σ : MΘ → AΘ [10] . An S-system AΘ is injective if and only if it is MΘ-injective for all S-systems MΘ . An S-system AΘ is quasi injective if and only if it is AΘ-injective . Quasi injective S-systems have been studied by Lopez and Luedeman [8] . It is clear that every injective system is quasi injective but the converse is not true in general see [8] . An S-system AΘ is weakly injective if it is injective relative to all embeddings of right ideals into SΘ , [7,p.205] .
In this work, we find weak form of quasi injectivity called finitely quasi injective and quasi finitely injective systems over monoids. Also, we give some interesting results on these systems.

II-FINITELY QUASI INJECTIVE SYSTEMS OVER MONOIDS

In [9], V.S.Ramamurthi define finitely injective module which motivate us to define finitely injective relative to S-system as follows:

Definition (2.1): Let \( M_2 \) and \( N_2 \) be two \( S \)-systems. \( M_2 \) is called \( \alpha \)-finitely \( N_2 \)-injective (for short \( FQ \)-injective) if every homomorphism from a finitely generated subsystem of \( N_2 \) to \( M_2 \) extends to homomorphism of \( N_2 \) into \( M_2 \). An \( S \)-system \( M_2 \) is called finitely quasi injective (for short \( FQ \)-injective) if \( M_2 \) is \( F \)-\( M \)-injective system.

Example and Remarks (2.2):

1. Every quasi injective systems is \( FQ \)-injective systems, but the converse is not true in general as the following example shows: let \( S \) be the monoid \( \{1,a,b,0\} \) with \( ab = a^2 = a \) and \( ba = b^2 = b \). Now, consider \( S \) as a right \( S \)-system over itself, then it is easy to check that \( S_1 \) is \( FQ \)-injective. But, when we take \( N = \{a,0\} \) be a subsystem of \( S_1 \) and \( f \) be \( S \)-homomorphism defined by \( f(x) = \begin{cases} 0 & \text{if} \ x = 0 \\ b & \text{if} \ x = a \end{cases} \), then this \( S \)-homomorphism cannot be extended to \( S \)-endomorphism of \( S_1 \). If not, that is there exists \( S \)-homomorphism \( g: S_1 \to S_1 \) such that \( g(x) = f(x) \) for each \( x \in N \), which is the trivial \( S \)-homomorphism, since other extension is not \( S \)-homomorphism. Then, \( b = f(a) = g(a) = a \) which implies that \( b = a \), and this is a contradiction.

2. Isomorphic system to \( F \)-\( M \)-injective is \( F \)-\( M \)-injective for any \( S \)-system \( M \). In particular, isomorphic system to \( FQ \)-injective is \( FQ \)-injective.

3. Let \( N_1 \) and \( N_2 \) be two \( S \)-systems such that \( N_1 \cong N_2 \). If \( M_2 \) is \( F \)-\( N_1 \)-injective, then \( M_2 \) is \( F \)-\( N_2 \)-injective.

In the following theorem, we give characterizations of \( FQ \)-injective \( S \)-systems:

Theorem (2.3): The following statements are equivalent for \( S \)-system \( M_2 \) with \( T = \text{End}_R(M_2) \):

1. \( M_2 \) is \( FQ \)-injective.
2. \( \gamma_{S_n}(x) \subseteq \gamma_{S_n}(y) \), where \( x, y \in M^n, n \in \mathbb{Z}^+ \) implies that \( Ty \subseteq Tx \).
3. If \( x \in M_2 \), \( i = 1, 2, \ldots, n \) and \( \alpha : \bigcup_{i=1}^n x_i S \to M_2 \) is \( S \)-homomorphism, then there exists \( S \)-homomorphism extends \( \alpha \).

Proof: Put \( M^n = M^{n \times n} \) and \( S_n = S_{n \times 1} \).

(1→2) Let \( \gamma_{S_n}(x) = \{ (s,s') \in S_n \mid xs = xs' \} \), where \( s = \begin{pmatrix} s_1' \\ \vdots \\ s_n' \end{pmatrix} \) and \( s' = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \) and \( \gamma_{S_n}(y) = \gamma_{S_n}(y) \) such that \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in M^n \), \( n \in \mathbb{Z}^+ \). Then, \( \alpha : \bigcup_{i=1}^n x_i S \to M_2 \) is defined by \( \alpha(xs) = ys \). It is obvious that \( \alpha \) is well-defined and \( S \)-homomorphism. Since \( M_2 \) is \( FQ \)-injective, so there exists \( \sigma \in T \) such that \( \sigma \) extends \( \alpha \), then \( y = \alpha(x_i) = \sigma(x_i) \), where \( i = 1, 2, \ldots, n \), so \( y = \sigma x \) and then \( Ty \subseteq Tx \).

(2→3) As \( \alpha \) is \( S \)-homomorphism and \( \beta \) is \( S \)-monomorphism, so we have \( \gamma_{S_n}(\beta(x_1), \ldots, \beta(x_n)) \subseteq \gamma_{S_n}(\alpha(x_1), \ldots, \alpha(x_n)) \) by (2), we have \( T\alpha(x) \subseteq T\beta(x) \), where \( \alpha(x) = (\alpha(x_1), \ldots, \alpha(x_n)) = (x_1, \ldots, x_n) \) and \( \beta(x) = (\beta(x_1), \ldots, \beta(x_n)) = \beta(x_1, \ldots, x_n) \). Thus there exists \( \sigma \in T \) such that \( (\alpha(x_1), \ldots, \alpha(x_n)) = \sigma(\beta(x_1), \ldots, \beta(x_n)) \), \( \sigma\alpha(x) = \sigma\beta(x) \). Therefore \( \alpha = \sigma\beta \).

(3→1) By definition of \( FQ \)-injective system.
Corollary (2.4) : The following statements are equivalent for a monoid $S$:

1. $S$ is a right F-injective.
2. $\gamma_S(\alpha) \subseteq \gamma_S(\beta)$, where $\alpha, \beta \in S^a$, $n \in \mathbb{Z}^+$ implies that $S\beta \subseteq S\alpha$.
3. If $a \in S$, $i = 1, 2, \ldots, n$ and $\alpha : \bigcup_{i=1}^n a_i S \rightarrow S$ is S-homomorphism, then there exists S-homomorphism $b$ belong to $S$ which is extends $\alpha$.

The following proposition gives a condition under which subsystem of FQ-injective inherit this property. Before this, we need the following concept:

Recall that a subsystem $N$ of $S$-system $M_s$ is fully invariant of $M_s$ if $f(N) \subseteq N$, for all $f \in \text{End}_s(M_s)$ [5]. An $S$-system is called duo if each subsystem of it is fully invariant.

Proposition (2.5) : Every fully invariant subsystem of FQ-injective system is FQ-injective.

Proof: Let $M_s$ be FQ-injective system and $N$ be a fully invariant subsystem of $M_s$. Let $X$ be any finitely generated subsystem of $N$ and $f$ be S-homomorphism from $X$ into $N$. Since $M_s$ is FQ-injective system, so there exists an $S$-endomorphism $g$ of $M_s$ such that $g_0i_0 = i_0f$, where $i_0$ and $i_0$ are the inclusion maps of $X$ into $N$ and $N$ into $M_s$ respectively. As $N$ is fully invariant in $M_s$, so $g(N) \subseteq N$. Put $g_1 = h$, then $\forall x \in X$, we have $(h \circ i_0)(x) = g(x) = (g_0 i_0)(x) = (i_0 f)(x)$. Therefore $N$ is FQ-injective system.

Recall that an $S$-system $M_s$ is called multiplication if every subsystem of $M_s$ is of the form $MI$ for some right ideal of $S$. It is clear that every subsystem of multiplication system is fully invariant [5].

Corollary (2.6) : If $M_s$ is FQ-injective duo (multiplication) system, then every subsystem of $M_s$ is FQ-injective.

Proposition (2.7) : Let $M_s$ and $N_s$ be two $S$-systems and $N'$ a subsystem of $N_s$. If $M_s$ is F-N-injective, then:

1. Every retract of $M_s$ is F-N-injective.
2. $M_s$ is F-N'-injective system.

Proof: (1) Let $M_s = M_s \oplus M_s$ and $K$ be a finitely generated subsystem of $N$ and $f$ be S-homomorphism of $K$ into $M_s$. Since $M_s$ is F-N-injective, so $j_0(f)$ which $j_0$ is injection of $M_s$ into $M_s$ extends to $S$-homomorphism $g$ of $N_s$ into $M_s$, such that $g_0 i_0 = j_0 f$. Put $g = \pi_1 g_j : N_s \rightarrow M_s$, where $\pi_1$ be the projection map of $M_s$ into $M_s$, then $g_0 i_0 = \pi_1 g_0 i_0 = \pi_1 j_0 f$. Thus $f$ extends to S-homomorphism $g$ and $M_s$ is F-N-injective system.

(2) It is obvious.

The following corollaries is immediately from above proposition:

Corollary (2.8): Retract of FQ-injective system is FQ-injective.

Corollary (2.9): Let $N$ be any subsystem of system $M_s$. If $N$ is F-M-injective, then $N$ is finitely injective.

Proposition (2.10) : Let $M_s$ and $N_s$ be two $S$-systems. Let $N_s$ be finitely generated subsystem of $M_s$. Then $N_s$ is F-M-injective if and only if every monomorphism $f : N_s \rightarrow M_s$ split.

Proof: Assume that $N_s$ is F-M$_s$-injective system and $f : N_s \rightarrow M_s$ be monomorphism, then by F-M$_s$-injective of $N_s$, there exists an S-homomorphism $g : M_s \rightarrow N_s$ such that $g f = I_{N_s}$. Since $N_s \supseteq f(N_s)$, so $f(N_s)$ be a retract of $M_s$. Conversely, assume that $A$ is finitely generated subsystem of $M_s$. Then, by assumption the monomorphism (inclusion map) $i_0$ of $A$ into $M_s$ split, this means there exists $\omega : M_s \rightarrow A$ such that $\omega i_0 = I_A$. Now, for S-homomorphism $f : A \rightarrow N_s$, set $g = (\omega g): M_s \rightarrow N_s$ which implies that $g_0 i_0 = f_0 (\omega g) = f_0 i_0 = f$. Thus $N_s$ is F-M-injective system.
Corollary (2.11): Let \( N_i \) be a finitely generated subsystem of an S-system \( M_i \). If \( N_i \) is F-\( M_i \)-injective system, then \( N_i \) is a retract of \( M_i \).

Corollary (2.12): Let \( M_i \) be FQ-injective S-system. Then, every finitely generated subsystem of \( M_i \) which is isomorphic to \( M_i \) is a retract of \( M_i \).

Definition (2.13): An S-system \( M_i \) is called FC\(_2\) if every finitely generated subsystem of \( M_i \) that is isomorphic to a retract of \( M_i \) is itself a retract of \( M_i \).

Theorem (2.14): Every FQ-injective system satisfies FC\(_2\).

Proof: Let \( M_i \) be FQ-injective S-system and \( A \) be a retract of \( M_i \), with \( A \cong B \), where \( B \) is finitely generated subsystem of \( M_i \). Let \( f \) be an S-isomorphism from \( B \) into \( A \), then \( f \) is S-monomorphism from \( B \) into \( A \). Since \( A \) is a retract of \( M_i \), so by corollary (2.8) \( A \) is F-Q-injective system. By example and remarks (2.2), since \( A \) is \( M \)-injective system. Then, by proposition (2.10) \( f \) is split and by corollary (2.9) \( B \) is a retract of \( M_i \), so \( M_i \) satisfies FC\(_2\) – condition.

Proposition (2.15): Let \( M_i \) be an S-system and \( \{N_i\}_{i \in I} \) be a family of S-systems, where \( I \) is finite index set. Then \( I_i \in I \) is finitely \( M_i \)-injective if and only if for each \( i \in I \), \( N_i \) is finitely M-injective system.

Proof: \( \Rightarrow \) Put \( N_i = \Pi_{i \in I} N_i \), assume that \( N_i \) is F-M-injective S-system and \( A \) is a finitely generated subsystem of \( M_i \). Let \( f \) be an S-homomorphism of \( A \) into \( N_i \). Since \( N_i \) is F-M-injective, so there exists S-homomorphism \( g : M_i \to N_i \) such that \( g \circ i_A = j \circ f \), where \( j \) is the injection map of \( N_i \) into \( N_i \) and \( i_A \) is the inclusion map of \( A \) into \( M_i \). Now, let \( \pi_i \) be the projection map of \( N \) onto \( N_i \). Put \( h(=\pi_i \circ g) : M_i \to N_i \), then \( \forall \; a \in A \), \( (h \circ j)(a) = (\pi_i \circ g)(a) = f(a) \). Thus \( N_i \) is F-M-injective system.

\( \Leftarrow \) Assume that \( N_i \) is F-M-injective for each \( i \in I \). Let \( A \) be a finitely generated subsystem of \( M_i \) and \( f \) be an S-homomorphism of \( A \) into \( N_i \). Since \( N_i \) is F-M-injective system, so there exists S-homomorphism \( \beta : M_i \to N_i \) such that \( \beta \circ i_A = \pi_i \circ f \), where \( i_A \) is the inclusion map of \( A \) into \( M_i \). Now, define an S-homomorphism \( \beta (\neq j \circ \beta) : M_i \to N_i \), then \( \beta \circ i_A = j \circ \beta \circ i_A = j \circ \pi_i \circ f \circ i_A = f \). Therefore \( N_i \) is F-M-injective system.

Corollary (2.16): Let \( M_i \) and \( N_i \) be S-systems, where \( i \in I \) and \( I \) is finite index set. If \( \bigoplus_{i \in I} N_i \) is F-M-injective for all \( i \in I \), then \( N_i \) is F-M-injective.

The following proposition give another characterization of FQ-injective S-system:

Proposition (2.17): If \( M_i \) is FQ-injective S-system and \( T = \text{End}(M_i) \), then \( TA = TB \) for each isomorphic subsystems \( A \) and \( B \) of \( M_i \).

Proof: By assumption there exists an S-isomorphism \( \alpha : A \to B \), let \( b \in B \) so there exists \( a \in A \) such that \( \alpha(a) = b \). For \( i \in S \), if \( a = \alpha \cdot \alpha \) and since \( \alpha \) is well-defined, so \( \alpha(a) = \alpha(\alpha) \), then \( b = \alpha \cdot \alpha \), which implies that \( \gamma_i(a) \subseteq \gamma_i(b) \). Since \( M_i \) is FQ-injective, then by theorem (2.3), \( T \subseteq T \) and hence \( T \subseteq T \) \( \forall \; b \in B \). Thus \( T \subseteq T \) \( \forall \; b \in B \). Similarly, we can prove \( T \subseteq T \). Therefore \( T \subseteq T \).

As an immediate consequence of above proposition, we have the following result:

Corollary (2.18): If \( S \) is F-injective monoid and \( A \), \( B \) are two isomorphic ideal of \( S \), then \( A = B \).

Recall that two S-systems \( M_i \) and \( N_i \) are mutually finitely injective if \( M_i \) is finitely \( N_i \)-injective and \( N_i \) is finitely M-injective.

Theorem (2.19): If \( M_i \oplus M_j \) is FQ-injective system, then \( M_i \) and \( M_j \) are mutually F-injective system.
The proof of the following corollary is immediately from above theorem and proposition (2.7):

**Corollary (2.20):** If $\bigoplus_{i \in I} M_i$ is FQ-injective system, then $M_i$ is F-M$_k$-injective for all distinct $j, k \in I$.

**Definition (2.21):** An S-system $M$ is called quasi finitely injective (for short QF-injective) if every S-homomorphism from a finitely $M_i$-generated subsystem of $M_i$ to $M$ extends to an S-endomorphism of $M$.

**Proposition (2.22):** The following statements are equivalent for S-system $M_i$ with $T = \text{End}_n(M_i)$:

1. $M_i$ is QF-injective.
2. $\gamma_{M_{\beta}}(a) \subseteq \gamma_{M_{\beta}}(\beta)$, where $a, \beta \in T^n$, $n \in \mathbb{Z}^+$ implies that $T\beta \subseteq T\alpha$.

**Proof:** (1$\to$2) Assume that $\gamma_{M_{\beta}}(a) \subseteq \gamma_{M_{\beta}}(\beta)$ such that $a, \beta \in T^n, n \in \mathbb{Z}^+$. Write $a = (a_1, \ldots, a_n), \beta = (\beta_1, \ldots, \beta_n)$, then the mapping $A: \bigcup_{i=1}^n a_i M_i \to M_i$ defined by $A(a_i) = \beta_i m_i$ is well-defined and S-homomorphism, for this let $a_i m_i = \beta_j k_i \forall i \in I$, so $(m_i, k_i) \in \gamma_{M_{\beta}}(a) \subseteq \gamma_{M_{\beta}}(\beta)$ which implies that $\beta_i m_i = \beta_j k_i$ and then $f(a_i m_i) = f(a_i k_i)$. Also, for S-homomorphism, we have $\gamma_{M_{\beta}}(a) = \gamma_{M_{\beta}}(\beta)$, which implies that $\beta_i m_i = \beta_j k_i$. Since $M_i$ is QF-injective, so there exists S-endomorphism $g$ of $M_i$ which extends $f$, then $\beta_i m_i = g(a_i m_i) = f(a_i m_i) \forall i \in I$ and $M_i \in M_i$. Thus $\beta = g\alpha$ and so $T \beta \subseteq T \alpha$.

(2$\to$1) Assume that $A: \bigcup_{i=1}^n a_i M_i \to M_i$ be homomorphism. Put $a = (a_1, \ldots, a_n), \beta = (f \beta_1, \ldots, f \beta_n)$, then it is easy to check that $\gamma_{M_{\beta}}(a) \subseteq \gamma_{M_{\beta}}(\beta)$. By (2), we have $\beta \in T \alpha$, so there exists $\sigma \in T$ such that $\beta = \sigma \alpha$. Since $f(a(M)) = \beta(M) = \sigma(a(M))$. Thus $\sigma$ extends $f$.

The following proposition gives a condition under which endomorphism of S-system is QF-injective:

**Proposition (2.23):** Given an S-system $M_i$ with $T = \text{End}_n(M_i)$. Let $a, \beta$ denote elements of $T$. Assume that $M_i \times M_i$ generates ker for each $a \in T$. Then $T$ is right QF-injective if and only if $\ker(a) \subseteq \ker(b)$ implies that $b \in T \alpha$.

**Proof:** If $T$ is right QF-injective, then the condition holds for any $M_i$. Conversely, if $\beta \in T \ker(\alpha) = T \alpha$, so there exists $\sigma \in T$ such that $\beta = \sigma \alpha$. The proof is complete when we prove $\ker(\alpha) \subseteq \ker(b)$. Since $M_i \times M_i$ generates ker $a$, so there exists S-epimorphism $f_i: M_i \times M_i \to \ker(\alpha)$ such that $\forall(x, y) \in \ker(\alpha)$, we have $a(x) = a(y)$, and then there exists $(m, k) \in M_i \times M_i$, where $x = f_m y = f_k$. Now, since $\sigma$ is well-defined, so $\sigma a(x) = \sigma a(y)$ which implies that $\beta(x) = \beta(y)$ and $(x, y) \in \ker(\alpha)$. Thus $T$ is QF-injective by proposition (2.22).

The following proposition gives a condition under which endomorphism of QF-injective system is F-injective:

**Proposition (2.24):** Let $M_i$ be a right $S$-system with $T = \text{End}_n(M_i)$, then:

1. If $T$ is right F-injective, then $M_i$ is QF-injective.
2. If $M_i$ is QF-injective and $M_i \times M_i$ generates $\gamma_{M_{\alpha}}(a)$ for any positive integer $n$ and $a \in T^n$, then $T$ is right F-injective.

**Proof:** (1) If $T$ is right F-injective, then $M_i$ is F-injective.

(2) If $M_i$ is QF-injective and $M_i \times M_i$ generates $\gamma_{M_{\alpha}}(a)$ for any positive integer $n$ and $a \in T^n$, then $T$ is right F-injective.

**Proof:** (1) Let $\gamma_{M_{\alpha}}(a) \subseteq \gamma_{M_{\beta}}(\beta)$, where $a, \beta \in T^n, n \in \mathbb{Z}^+$, then $\gamma_{T^n}(a) \subseteq \gamma_{T^n}(\beta)$. Since $T$ is right F-injective, so by corollary (2.4) we have $T \beta \subseteq T \alpha$. Then, by proposition (2.26) $M_i$ is QF-injective system.

(2) Let $\gamma_{T^n}(a) \subseteq \gamma_{T^n}(\beta)$, where $a, \beta \in T^n, n \in \mathbb{Z}^+$. Then, for any $(x, y) \in \gamma_{M_{\alpha}}(a)$, we have $a(x) = a(y)$. Since $M_i \times M_i$ generates $\gamma_{M_{\alpha}}(a)$, so $x = \lambda m, y = \lambda k$, where $(m, k) \in M_i \times M_i$ and $\lambda \in T^n$. Then, $(\lambda m, \lambda k) \in \gamma_{T^n}(a) \subseteq \gamma_{T^n}(\beta)$, so
\[ \beta(\lambda, m) = \beta(\lambda, k). \] This means that \[ \beta(x) = \beta(y) \text{ and } (x, y) \in \gamma_{M^s}(\beta). \] Hence \[ \gamma_{M^s}(\alpha) \subseteq \gamma_{M^s}(\beta). \] Since \( M_s \) is QF-injective system, so \( T\beta \subseteq T\alpha \) and consequently, \( T \) is F-injective by corollary (2.4).

### III - RELATIONSHIP AMONG FQ-INJECTIVE AND QF-INJECTIVE S-SYSTEMS WITH OTHER CLASSES OF INJECTIVITY

The following proposition gives a condition under which FQ-injective system is QF-injective system, but before this we need the following concept:

**Definition (3.1):** An S-system \( M_s \) is called self-generator if it generates all its subsystems.

**Proposition (3.2):** If \( M_s \) is finitely generated S-system which is self-generator, then \( M_s \) is FQ-injective system if and only if \( M_s \) is QF-injective system.

**Proof:** Assume that \( M_s \) is QF-injective system. Let \( X \) be finitely \( M_s \)-generated subsystem of \( M_s \) and \( f \) be \( S \)-homomorphism of \( X \) into \( M_s \). Since \( M_s \) is finitely generated and \( X \) is finitely \( M_s \)-generated, so there exists \( S \)-epimorphism \( \alpha: X \rightarrow M_s \). Since \( M_s \) is QF-injective system, so \( f \) extends to \( S \)-endomorphism \( g \) of \( M_s \) such that \( g_{|\alpha} = f \), where \( g_{|\alpha} \) is the inclusion map of \( X \) into \( M_s \) and then \( M_s \) is QF-injective system. Conversely, assume that \( M_s \) is QF-injective system. Let \( A \) be finitely generated system of \( M_s \) and \( f \) be \( S \)-homomorphism of \( A \) into \( M_s \). Since \( M_s \) is self-generator, so there exists \( S \)-epimorphism \( \alpha: M_s \rightarrow A \), and then \( A \) is finitely \( M_s \)-generated. Since \( M_s \) is QF-injective system, so \( f \) extends to \( S \)-endomorphism \( g \) of \( M_s \) such that \( g_{|\alpha} = f \), where \( g_{|\alpha} \) is the inclusion map of \( A \) into \( M_s \) and then \( M_s \) is QF-injective system.

The following proposition explains under which condition on finitely E(\( M_s \))-injective to be injective:

**Proposition (3.3):** Let \( M_s \) be a finitely generated S-system. Then \( M_s \) is injective system if and only if \( M_s \) is finitely E(\( M_s \))-injective.

**Proof:** (\( \Rightarrow \)) It is obvious.

(\( \Leftarrow \)) Let \( M_s \) be finitely E(\( M_s \))-injective and \( f \) be \( S \)-monomorphism from \( M_s \) into E(\( M_s \)). Since \( M_s \) is finitely E(\( M_s \))-injective, so by proposition (2.10), there exists an \( S \)-homomorphism \( g: E( M_s ) \rightarrow M_s \) such that \( g f = 1_{ M_s } \) which implies that \( f \) is split and \( f(M_s) \) is retract of E(\( M_s \)). Since \( M_s \) is QF-injective system, \( f(M_s) \) is injective, so \( M_s \) is injective.

As a particular case of above proposition, we have the following corollary:

**Corollary (3.4):** A monoid \( S \) is self-injective if and only if \( S \) is finitely \( S \)-injective S-system.

The following proposition explains under which condition on FQ-injective to being injective, but before this we need the following concept:

**Definition (3.6):** An S-system \( M_s \) is said to be weakly injective if for every finitely generated subsystem \( N \) of \( E( M_s ) \), we have \( N \subseteq X \subseteq E( M_s ) \) for some \( X \cong M_s \).

**Proposition (3.7):** Let \( M_s \) be a finitely generated system. Then \( M_s \) is injective system if and only if \( M_s \) is weakly injective and FQ-injective.

**Proof:** (\( \Rightarrow \)) It is obvious.

(\( \Leftarrow \)) It is enough to prove that \( M_s = E( M_s ) \). Let \( x \in E( M_s ) \), so \( M_s \cup xS \) is finitely generated. As \( M_s \) is weakly injective, so there exists subsystem \( X \) of \( E( M_s ) \) such that \( M_s \cup X \cong X \cong M_s \). Since \( M_s \) is FQ-injective system, so \( X \) is also FQ-injective by example and remarks (2.2)(2). By theorem (2.14), \( X \) is satisfy FC and since \( M_s \) is finitely generated subsystem of \( X \), \( M_s \) is a retract of \( X \). But \( M_s \) is \( \cap \)-large subsystem of \( E( M_s ) \), so \( M_s \) is \( \cap \)-large in \( X \). Therefore \( M_s = X \) and \( x \in M_s \). Thus, \( M_s = E( M_s ) \) is injective.
REFERENCES


