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Representation of Numerical Data by Makeham's Curve and Third Degree Polynomial Curve: A Comparative Study

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ABSTRACT: Some methods have already been developed for representing a set of numerical data on a pair of variables by some mathematical curves namely polynomial curve, exponential curve, modified exponential curve and logistic curve. A set of available numerical data on a pair of variables are required to be represented by different types of mathematical curves in different situations. In some situation, it is required to represent the set of numerical data by Makeham's curve. The mathematical function of Makeham's curve contains four parameters. On the other hand, the mathematical function of a third degree polynomial curve contains three parameters. Therefore, a comparative study has been carried out on the degree of suitability of these two types of curves in the case of total population of India & of Assam. This paper is based on the findings of this study.

KEYWORDS: Pair of variables, numerical data, Makeham's curve, third degree polynomial curve, Population, India, Assam.

I. INTRODUCTION

The existing formulae for numerical interpolation {Hummel (1947), Erdos & Turan (1938), Bathe & Wilson (1976), Jan (1930), Hummel (1947) et al} carry the task of repetitive computations i.e. if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. In order to get rid of these repeated numerical computations from the given data, one can think of an approach which consists of the representation of the given numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. However, a method is necessary for representing a given set of numerical data on a pair of variables by a suitable polynomial. Das & Chakrabarty (2016*a*, 2016*b*, 2016*c* & 2016*d*) derived four formulae for representing numerical data on a pair of variables by a polynomial curve. They derived the formulae from Lagranges Interpolation Formula, Newton's Divided Difference Interpolation Formula, Newton's Forward Interpolation Formula and Newton's Backward Interpolation respectively. In another study, Das & Chakrabarty (2016*e*) derived one method for representing numerical data on a pair of variables by a polynomial curve. The method derived is based on the inversion of a square matrix by Caley-Hamilton theorem on characteristic equation of matrix *[Cayley (1858 , 1889) & Hamilton (1864a , 1864b , 1862)]*. In a separate study Das & Chakrabarty (2016*f* , 2016*g*) composed two methods, based on the inversion of matrix by elementary transformation, for the same purpose. The studies, made so far, are on the representation of numerical data on a pair of variables by polynomial curve It is be possible to represent the numerical data on a pair of variables by non-polynomial curve besides the representation of the said data by polynomial curve. Some methods have already been developed for representing the said data by some mathematical curves namely polynomial curve [Das & Chakrabarty (2016*a*, 2016*b* , 2016*c* , 2016*d* , 2016*e* , 2016*f* , 2017*a*, 2017*b*)], exponential curve [Das & Chakrabarty (2017*c*)] and modified exponential curve [Das & Chakrabarty (2017*d*)]. A set of available numerical data on a pair of variables are required to be represented by different types of mathematical curves in different situations. In some situation, it is required to represent the data as mentioned by a Makeham's curve. In this study, attempt has been made on the representation of numerical data by this mathematical curve. In this connection, it is to be noted that the mathematical function of the Makeham's curve contains four parameters. On the other hand, the mathematical function of a third degree polynomial curve contains three parameters. Therefore, a comparative study has been carried out on the degree of suitability of these two types of curves in the case of total population of India & of Assam. This paper is based on the findings of this study.

II. REPRESENTATION OF NUMERICAL DATA BY MAKEHAM'S CURVE

The Makeham's curve is of the form

$$y = a\, b^x c^{d^x} \tag{2.1}$$

where a , b, c and d are parameters

Let

$$y_0\ ,\ y_1\ ,\ y_2\ ,\ y_3$$

be the values of y corresponding to the values of x_0 , x_1 , x_2 and x_3 of x respectively.

Then the points

$$(x_0, y_0)\ , (x_1, y_1)\ , (x_2, y_2) \text{ and } (x_3, y_3)$$

lie on the curve described by equation (2.1).
Therefore,

$$y_0 = a\, b^{x_0}\, c^{d^{x_0}} \tag{2.2}$$
$$y_1 = a\, b^{x_1}\, c^{d^{x_1}} \tag{2.3}$$
$$y_2 = a\, b^{x_2}\, c^{d^{x_2}} \tag{2.4}$$
$$y_3 = a\, b^{x_3}\, c^{d^{x_3}} \tag{2.5}$$

$$\therefore\ (2.2) \Rightarrow \log y_0 = \log a + x_0 \log b + d^{x_0} \log c \tag{2.6}$$

$$(2.3) \Rightarrow \log y_1 = \log a + x_1 \log b + d^{x_1} \log c \tag{2.7}$$

$$(2.4) \Rightarrow \log y_2 = \log a + x_2 \log b + d^{x_2} \log c \tag{2.8}$$

$$(2.5) \Rightarrow \log y_3 = \log a + x_3 \log b + d^{x_3} \log c \tag{2.9}$$

If x_0 , x_1 , x_2 , x_3 are equally spaced then

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$$

i.e. $x_1 = x_0 + h$ $\quad x_2 = x_0 + 2h$ and $x_3 = x_0 + 3h$

Equation (2.7) – Equation (2.6) $\Rightarrow \log y_1 - \log y_0 = (x_1 - x_0) \log b + (d^h - 1) \log c\, d^{x_0}$

$$\Rightarrow \Delta \log y_0 = h \log b + (d^h - 1) \log c\, d^{x_0} \tag{2.10}$$

Equation (2.8) – Equation (2.7) $\Rightarrow \log y_2 - \log y_1 = (x_2 - x_1) \log b + (d^h - 1) \log c\, d^{x_1}$

$$\Rightarrow \Delta \log y_1 = h \log b + (d^h - 1) \log c\, d^{x_1} \tag{2.11}$$

Equation (2.9) – Equation (2.8) $\Rightarrow \log y_3 - \log y_2 = (x_3 - x_2) \log b + (d^h - 1) \log c\, d^{x_2}$

$$\Rightarrow \Delta \log y_2 = h \log b + (d^h - 1) \log c\, d^{x_2} \tag{2.12}$$

Again ,

Equation (2.11) – Equation (2.10) $\Rightarrow \Delta \log y_1 - \Delta \log y_0 = (d^h - 1) \log c\, d^{x_1} - (d^h - 1) \log c\, d^{x_0}$

$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1) \log c\, (d^{x_1} - d^{x_0})$$

$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1) \log c\, (d^{x_0 + h} - d^{x_0})$$

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$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1) \log c \,(d^h - 1)\, d^{x_0}$$

$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1)^2 \log c \, d^{x_0} \qquad (2.13)$$

Similarly,

$$\text{Equation (2.12)} - \text{Equation (2.11)} \Rightarrow \Delta^2 \log y_1 = (d^h - 1)^2 \log c \, d^{x_1} \qquad (2.14)$$

$$\therefore \frac{(2.14)}{(2.13)} \Rightarrow \frac{\Delta^2 \log y_1}{\Delta^2 \log y_0} = \frac{d^{x_1}}{d^{x_0}} = \frac{d^{x_0+h}}{d^{x_0}} = \frac{d^{x_0}.d^h}{d^{x_0}} = d^h \qquad (2.15)$$

If $h = 1$, $\boldsymbol{d} = \frac{\Delta^2 \log y_1}{\Delta^2 \log y_0}$

$\therefore$ (2.13) $\Rightarrow \Delta^2 \log y_0 = (d-1)^2 \log c$ (If $x_0 = 0$)

$$\Rightarrow \log c = \frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \boldsymbol{c = antilog\left\{ \frac{\Delta^2 \log y_0}{(d-1)^2} \right\}} \qquad (2.16)$$

(2.10) $\Rightarrow \Delta \log y_0 = \log b + (d-1) \log c$

$$\Rightarrow \Delta \log y_0 = \log b + (d-1)\,.\,\frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \Delta \log y_0 = \log b + \frac{\Delta^2 \log y_0}{(d-1)}$$

$$\Rightarrow \log b = \Delta \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)}$$

$$\Rightarrow \boldsymbol{b = antilog\left[\Delta \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)}\right]} \qquad (2.17)$$

(2.6) $\Rightarrow \log y_0 = \log a + x_0 \log b + \log c$

$$\Rightarrow \log y_0 = \log a + \log c$$

$$\Rightarrow \log a = \log y_0 - \log c$$

$$\Rightarrow \log a = \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \boldsymbol{a = antilog\left\{\log y_0 - \frac{\Delta^2 \log y_0}{(d-1)^2}\right\}} \qquad (2.18)$$

Note: It is to be noted that the parameters ***a***, ***b***, ***c*** & ***d*** are to computed in the following order:
Step-1: Compute parameter *a* by equation (2.18).
Step-2: Compute parameter *b* by equation (2.17) substituting the value of *a* obtained in Step-1.
Step-3: Compute parameter *c* by equation (2.16) substituting the values of *a* & *b* obtained in Step-1.& Step-2 respectively.
Step-4: Compute parameter *d* by equation (2.15) substituting the values of *a*, *b* & *c* obtained in Step-1, Step-2 & Step-3 respectively.

III. MAKEHAM'S CURVE & CUBIC CURVE – COMPARISON OF SUITABILITY TO NUMERICAL DATA

The Makeham's curve contains four parameters. Accordingly, when four pairs of numerical data are available, they can be represented by Makeham's curve. However, when four pairs of numerical data are available then they can also be represented by cubic curve since this curve also contains four parameters. Thus, one question arises- which of the two curves will suit the entire data better between the two ones.

The cubic curve is of the form

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta \qquad (3.1)$$

where α , β, γ, δ are the parameters of the curve.

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In order to represent the given three pairs on numerical data by this curve, we have

$$y_0 = \alpha\, x_0^3 + \beta x_0^2 + \gamma x_0 + \delta \tag{3.2}$$
$$y_1 = \alpha\,(x_0 + h)^3 + \beta(x_0 + h)^2 + \gamma\,(x_0 + h) + \delta \tag{3.3}$$
$$y_2 = \alpha\,(x_0 + 2h)^3 + \beta(x_0 + 2h)^2 + \gamma\,(x_0 + 2h) + \delta \tag{3.4}$$
$$y_3 = \alpha\,(x_0 + 3h)^3 + \beta(x_0 + 3h)^2 + \gamma\,(x_0 + 3h) + \delta \tag{3.5}$$

The cubic curve described by equation (3.1) satisfies the property that

$$\Delta^3 y_i = \text{constant} \tag{3.6}$$

On the other hand, Makeham's curve described by the equation (2.1) satisfies the property that

$$\frac{\Delta^2 \log y_{i+1}}{\Delta^2 \log y_i} = \text{constant} \tag{3.7}$$

Thus, comparing the values of

$$\Delta^3 y_i \quad \& \quad \frac{\Delta^2 \log y_{i+1}}{\Delta^2 \log y_i}$$

one can determine which of the two curves will suit the entire data better between the two ones.

IV. APPLICATION IN DATA ON TOTAL POPULATION OF INDIA & OF ASSAM:

A .SUITABILITY OF CURVE IN THE CASE OF TOTAL POPULATION OF INDIA:

The following table shows the data on total population of India corresponding to the years:

Table - IV.A.a

(Total Population of India)

Year	1951	1961	1971	1981	1991	2001	2011
Total Population	361088090	439234771	548159652	683329097	846302688	1027015247	1210193422

Since four parameters are to be determined, it is required to select four equally spaced values of year.

Thus, these four years will be

1951, 1971, 1991, 2011

Accordingly, taking 1951 as origin and changing scale by 1/20, the following table can be obtained:

Table - IV.A.b

Year(t)	1951	1961	1971	1981	1991	2001	2011
x = (1/20).(t -1951)	0	0.5	1	1.5	2	2.5	3
$f(x)$	361088090	439234771	548159652	683329097	846302688	1027015247	1210193422

In order to examine the suitability of Makeham's curve & cubic curve, Table - IV.A.c & Table - IV.A.d have been constructer.

In Table - IV.A.c & Table- IV.A.d, it is observed the constancy property of $\frac{\Delta^2 \log y_{i+1}}{\Delta^2 \log y_i}$ is most dominant than that of $\Delta^3 y_i$. Thus, Makeham's curve is most suitable than cubic curve to represent the data on total population of India since 1951.

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Table - IV.A.c

Year	X	Total population (y_i)	$\Delta\, y_i$	$\Delta^2 y_i$	$\Delta^3 y_i$
1951	0	361088090			
			78146681		
1961	1	439234771		30778200	
			108924881		$-$ 4533636
1971	2	548159652		26244564	
			135169445		1559582
1981	3	683329097		27804146	
			162973591		$-$10065178
1991	4	846302688		17738968	
			180712559		$-$15273352
2001	5	1027015247		2465616	
			183178175		
2011	6	1210193422			
Total	21	5115322967	849105332		

Table - IV.A.d

Year	Total population (y_i)	$\boldsymbol{log\ y_i}$	$\boldsymbol{\Delta log\ y_i}$
1951	361088090	19.704632503150440464018364580816	
			0.195912110818992796522209 22265
1961	439234771	19.900544613969433260540573803466	
			0.22153252427925605890821620498
1971	548159652	20.122077138248689319448790008446	
			0.22041000364851677864518901799
1981	683329097	20.342487141897206098093979026436	
			0.213900498916744305347264289449
1991	846302688	20.556387640813950403441243315885	
			0.193534973122525033453744307274
2001	1027015247	20.749922613936475436894987623159	
			0.164123422736035200236347592146
2011	1210193422	20.914046036672510637131335215305	
Total	5115322967		

Table - IV.A.d Continued

Year	$\boldsymbol{\Delta^2\ log\ y_i}$	$\boldsymbol{\dfrac{\Delta^2\ log\ y_{i+1}}{\Delta^2\ log\ y_i}}$
1951		
1961	0.025620413460263262386006 98233	
		$-$ 0.043813525198579906518184340008 19

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1971	– 0.00112252063073928026302718699	5.7990067652345792978661532145229
1981	– 0.006509504731772473297924728541	3.1285829926225342920605116690177
1991	– 0.020365525794219271893519982175	1.4441832086082581615597161703986
2001	– 0.029411550386489833217396715128	
2011		

B. SUITABILITY OF CURVE IN THE CASE OF TOTAL POPULATION OF ASSAM:

The following table shows the data on total population of Assam corresponding to the years:

Table - IV.B.a
(Total Population of Assam)

Year	1951	1961	1971	1981	1991	2001	2011
Total Population	8028856	10837329	14625152	18041248	22414322	26638407	31205576

Since four parameters are to be determined, it is required to select four equally spaced values of year.
Thus, these four years will be

1951, 1971, 1991, 2011

Accordingly, taking 1951 as origin and changing scale by 1/20, the following table can be obtained:

Table - IV.B.b

Year(t)	1951	1961	1971	1981	1991	2001	2011
x = (1/20). (t -1951)	0	0.5	1	1.5	2	2.5	3
$f(x)$	8028856	10837329	14625152	18041248	22414322	26638407	31205576

In order to examine the suitability of Makeham's curve & cubic curve to the above data, Table - IV.B.c & Table-IV.B.d have been constructer.

Table - IV.B.c

Year	X	Total population (y_i)	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
1951	0	8028856			
			2808473		
1961	1	10837329		979350	
			3787823		– 1351077
1971	2	14625152		– 371727	
			3416096		1328705
1981	3	18041248		956978	
			4373074		–1105967
1991	4	22414322		–148989	

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			4224085		492073
2001	5	26638407		343084	
			4567169		
2011	6	31205576			
Total	21	5115322967	849105332		

Table - IV.B.d

Year	Total population (y_i)	$\boldsymbol{log\ y_i}$	$\boldsymbol{\Delta log\ y_i}$
1951	8028856	15.898552610020310347642892883553	
			0.299954511379428615699634498685
1961	10837329	16.198507121399738963342527382238	
			0.299746222790839955698223064354
1971	14625152	16.498253344190578919040750446592	
			0.209917905616261466899601251616
1981	18041248	16.708171249806840385940351698208	
			0.217039437705316399012031678624
1991	22414322	16.925210687512156784952383376832	
			0.172653917039078751272500644429
2001	26638407	17.097864604551235536224884021261	
			0.158242750209043092995689111694
2011	31205576	17.256107354760278629220573132955	
Total	5115322967		

Table - IV.B.d Continued

Year	$\boldsymbol{\Delta^2 log\ y_i}$	$\boldsymbol{\frac{\Delta^2 log\ y_{i+1}}{\Delta^2 log\ y_i}}$
1951		
1961	– 0.000208288588588660001411434331	
		431.268548043102316766687036937 25
1971	– 0.089828317174578488798621812738	
		– 0.079279366607908952553455518446 65
1981	0.007121532089054932112430427008	
		– 6.232580308730710114094407383181 6
1991	– 0.044385520666237647739531034195	
		0.324681711822244694402925619653 73
2001	– 0.014411166830035658276811532735	
2011		

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In Table- IV.B.c & Table - IV.B.d, it is observed the constancy property of both of $\frac{\Delta^2 \log y_{i+1}}{\Delta^2 \log y_i}$ and y_i are not satisfactory. Thus, neither of the two curves namely Makeham's curve nor cubic curve is satisfactorily suitable to represent the data on total population of Assam since 1951.

C. REPRESENTATION OF TOTAL POPULATION OF INDIA BY MAKEHAM'S CURVE:

Applying the formulas (2.15), (2.16), (2.17) & (2.18) it can be obtained that

$$\begin{aligned} d &= -4.5448066550163454958848230634923 \\ c &= 1.0005487253144715920892566206466 \\ b &= 1.5227019818642108469676443063929 \\ a &= 360890060.48806901244027752214465 \end{aligned}$$

Thus the mathematical form of Makeham's curve representing total population of India becomes

$$y = (360890060.48806901244027752214465).\times (1.5227019818642108469676443063929)^{x} \times (1.0005487253144715920892566206466)^{(-4.5448066550163454958848230634923\ )^{x}}$$

which is equivalent to

$$y = (360890060.48806901244027752214465)\times(1.5227019818642108469676443063929)^{\frac{t-1951}{20}} \times (1.0005487253144715920892566206466)^{(-4.5448066550163454958848230634923\ )^{\frac{t-1951}{20}}}$$

D. REPRESENTATION OF TOTAL POPULATION OF ASSAM BY MAKEHAM'S CURVE:

Applying the formulas (2.15), (2.16), (2.17) & (2.18) it can be obtained that

$$\begin{aligned} d &= 431.26854804310231676668703693725 \\ c &= 0.99999999887491256404961829679604 \\ b &= 1.3497980591806632796684482158043 \\ a &= 8028856.0090331650208179383118444 \end{aligned}$$

Thus the mathematical form of Makeham's curve representing total population of Assam becomes

$$y = (8028856.0090331650208179383118444) \times (1.3497980591806632796684482158043)^{x} \times (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725\ )^{x}}$$

which is equivalent to

$$y = (8028856.0090331650208179383118444) \times (1.3497980591806632796684482158043)^{\frac{t-1951}{20}} \times (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725\ )^{\frac{t-1951}{20}}}$$

V. CONCLUSION

It is to be mentioned here that the method discussed here is applicable in representing a set of numerical data on a pair of variables if the given values of the independent variable are equidistant. In the case where the given values of the independent variable are not equidistant, the method fails to represent the given data by makeham's curve.

Makeham's curve contains four parameters. Accordingly, when four pairs of numerical data are available, they can be represented by makeham curve.

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There may exists other form of curves containing four parameters. Representation of numerical data by these curves are yet to be studied.

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