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On the Oscillation of Nonlinear Fractional Differential Systems

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ABSTRACT: In this article, we obtain the oscillatory behavior of the solutions of fractional order differential system of the form

$$D_+^{\alpha}(u(t)) = a(t)f(v(t)),$$
$$D_+^{\alpha}(v(t)) = -b(t)g\left(\int_0^t (t-s)^{-\alpha} u(s)ds\right), t \geq t_0$$

where $0 < \alpha < 1$. By using generalized Riccati transformation and inequality technique, we will establish sufficient conditions for the oscillation of solutions of given system. Examples are given to illustrate the main results.

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I. INTRODUCTION

As the qualitative theory of nonlinear systems of differential equations originated by Henri Poincare at the end of nineteenth century. The study of oscillation of differential equations is one of the traditional trends in the qualitative theory of differential equations.

In the recent years, the number of investigations devoted to the oscillation theory of the functional differential equations has increased considerably, see the monographs [2, 6, 7, 14, 22], ordinary differential system have been studied by many authors [11, 15, 16, 18, 25] by different methods.

During the last two decades, fractional differential equations have progressively fascinated by many authors. The budding interesting in fractional differential equations is due to its effectiveness and applicability to miscellaneous branches of science and engineering. A rigorous and encyclopedic study of fractional differential equations can be found in [1, 4, 5, 10, 12, 20, 23, 24, 26] and in papers [3, 8, 9, 17, 21].

In [19], Lomtatidze et al. studied the oscillation and nonoscillation of two-dimensional linear differential systems of the form

$$u'(t) = q(t)v(t),$$
$$v'(t) = -p(t)u(t),\ t \geq t_{0.}$$

In [13], Philos et al. investigated the oscillation of nonlinear two-dimensional differential systems of the form

$$x'(t) = b(t)g(y(t)),$$
$$y'(t) = -a(t)f(x(t)), t \geq t_{0.}$$

To the best of the authors knowledge, no study has concerned with the oscillation of linear system of fractional order. In particular, it seems that there has been no work done on nonlinear fractional differential system. Motivated by the above observation, we propose system (1.1).

In the present paper, we consider the oscillatory behavior of the solutions of fractional order differential system of the form

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$$D_+^\alpha(u(t)) = a(t)f(v(t)),$$
$$D_+^\alpha(v(t)) = -b(t)g\left(\int_0^t (t-s)^{-\alpha} u(s)ds\right), t \ge t_0 \quad (1.1)$$

where $0 < \alpha < 1$, D_+^α denotes the Riemann-Liouville fractional derivative of order α with respect to t.

Throughout this paper, we assume that the following conditions:
(A_1) $a(t) \in C^\alpha([t_0,\infty),(0,\infty))$, $a(t) > 0$, with the condition $\int_{t_0}^\infty a(t)dt = \infty$;
(A_2) $b(t) \in C([t_0,\infty),[0,\infty))$ and $b(t)$ is not identically zero on any interval of the form $[\tau_0,\infty)$, where $\tau_0 \ge t_0$;
(A_3) $g \in C^1(R,R)$, $ug(u) > 0$, $g'(K(t)) \ge M > 0$ for $u \ne 0$, where $K(t) = \int_0^t (t-s)^{-\alpha} u(s)ds$;
(A_4) $f \in C^1(R,R)$, $vf(v) > 0$ and $f'(v(t)) \ge m > 0$ for $v \ne 0$;

A solution (u(t), v(t)) to the system (1.1) is oscillatory if it has arbitrarily large zeros, and is nonoscillatory otherwise. System (1.1) is said to be oscillatory if all its solutions are oscillatory otherwise it is nonoscillatory.

The main aim of this paper is to present some new oscillation criteria for the system (1.1) by using generalized Riccati transformation and inequality technique.

II. PRELIMINARIES

In this section, we give the definitions of fractional derivatives, integrals and lemmas which are useful throughout this paper. The following notations will be used for the convenience.

$$\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}, \xi_i = \frac{t_i{}^\alpha}{\Gamma(1+\alpha)}, i = 0,1,$$
$$p(t) = \tilde{p}(\xi), w(t) = \tilde{w}(\xi), a(t) = \tilde{a}(\xi), b(t) = \tilde{b}(\xi), q(t) = \tilde{q}(\xi), \rho(t) = \tilde{\rho}(\xi), \phi(t) = \tilde{\phi}(\xi), \delta(t) = \tilde{\delta}(\xi).$$

Definition: 2.1 [12] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y: R_+ \to R$ on the half-axis R_+ is given by

$$I_+^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} y(v)dv \text{ for } t > 0 \quad (2.1)$$

provided the right hand side is pointwise defined on R_+, where Γ is the gamma function.

Definition: 2.2 [12] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y: R_+ \to R$ on the half-axis R_+ is given by

$$D_+^\alpha y(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \left(I_+^{\lceil\alpha\rceil-\alpha} y\right)(t) > 0 \text{ for } t > 0 \quad (2.2)$$

provided the right hand side is pointwise defined on R_+, where $\lceil\alpha\rceil$ is the ceiling function of α.

Lemma: 2.1 [12] Let

$$K(t) = \int_0^t (t-s)^{-\alpha} y(s)ds \text{ for } \alpha \in (0,1) \text{ and } t > 0. \quad (2.3)$$

Then $K'(t) = \Gamma(1-\alpha) D_+^\alpha y(t)$.

III. MAIN RESULTS

In this section, we study oscillatory behaviour of (1.1) under certain conditions. We begin with the following lemma.

Lemma: 3.1. If (u(t),v(t)) is a nonoscillatory solution of (1.1), then u(t) is always nonoscillatory.

Proof. The proof follows from the line of Lemma 1.1in [15].

Theorem: 3.1 Suppose that the assumptions $(A_1) - (A_4)$ hold. Assume also that there exists a positive function $\delta \in C^\alpha([0,\infty); R_+))$ such that

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$$\limsup_{\xi\to\infty}\int_{\xi_0}^{\xi}\left[m\tilde{b}(s)\tilde{\delta}(s)-\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\left(\tilde{\delta}'(s)\right)^2}{\tilde{\delta}(s)}\right]ds=\infty. \tag{3.1}$$

Then every solution of system (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution of (u(t),v(t)) on $[t_0,\infty)$. From Lemma 3.1, u(t) is always nonoscillatory. Without loss of generality, we may assume that u(t) > 0 for $t \geq t_1 \geq t_0$ and K(t) > 0 for $t \geq t_1$. Since the similar argument holds for the case u(t) < 0 eventually. Then (1.1) can be reduced to the following inequality,

$$D_+^\alpha\left(\frac{1}{a(t)}D_+^\alpha u(t)\right)+mb(t)g\big(K(t)\big)\leq 0,\ t\geq t_1, \tag{3.2}$$

which implies that

$$D_+^\alpha\left(\frac{1}{a(t)}D_+^\alpha u(t)\right)<0,\ t\geq t_1. \tag{3.3}$$

Thus $D_+^\alpha u(t)\geq 0$ or $D_+^\alpha u(t)<0$, $t\geq t_1$. We claim that $D_+^\alpha u(t)\geq 0$ for $t\geq t_1$. Suppose not, there exists $T\geq t_1$ such that $D_+^\alpha u(T)<0$. Since $D_+^\alpha\left(\frac{1}{a(t)}D_+^\alpha u(t)\right)<0,\ t\geq t_1$. It is clear that $\frac{1}{a(t)}D_+^\alpha u(t)<\frac{1}{a(T)}D_+^\alpha u(T)$ for $t\geq T$. Therefore, by Lemma 2.1, we have

$$\frac{K'(t)}{\Gamma(1-\alpha)}=D_+^\alpha u(t)<\frac{a(t)}{a(T)}D_+^\alpha u(T).$$

Integrating the above inequality from T to t, we have

$$K(t)=K(T)+\Gamma(1-\alpha)\frac{D_+^\alpha u(T)}{a(T)}\int_T^t a(s)ds.$$

Letting t→ ∞, we get $\lim_{t\to\infty}K(t)\leq-\infty$. Which contradicts the fact that $D_+^\alpha u(t)\geq 0$ for $t\geq t_1$.

Define the function $w(t)$ by the generalized Riccati substititution

$$w(t)=\delta(t)\frac{D_+^\alpha u(t)}{a(t)g\big(K(t)\big)},\ t\geq t_1. \tag{3.4}$$

Then $w(t)>0$ for $t\geq t_1$. Differentiating (3.4) with respect to α, and using (3.1), (A_3), we have

$$D_+^\alpha w(t)\leq\frac{D_+^\alpha\delta(t)}{\delta(t)}w(t)-mb(t)\delta(t)-M\delta(t)\frac{D_+^\alpha u(t)}{a(t)}\frac{D_+^\alpha K(t)}{g^2(K(t))}.$$

Let $\delta(t)=\tilde{\delta}(\xi),\ a(t)=\tilde{a}(\xi),\ b(t)=\tilde{b}(\xi)\ and\ w(t)=\tilde{w}(\xi)$. Then $D_+^\alpha w(t)=\tilde{w}'(\xi)$, $D_+^\alpha\delta(t)=\tilde{\delta}'(\xi)$, $D_+^\alpha K(t)=\tilde{K}'(\xi)$, $D_+^\alpha u(t)=\tilde{u}'(\xi)$, and g(K(t)) = g($\tilde{K}(\xi)$).

Using these transformations in the last inequality, by Lemma 2.1, we have

$$\tilde{w}'(\xi)\leq\frac{\tilde{\delta}'(\xi)}{\tilde{\delta}(\xi)}\tilde{w}(\xi)-m\tilde{b}(\xi)\tilde{\delta}(\xi)-M\Gamma(1-\alpha)\frac{\tilde{a}(\xi)}{\tilde{\delta}(\xi)}\tilde{w}^2(\xi). \tag{3.5}$$

Then, with the inequality, $(\lambda\geq 1)$

$$\lambda AB^{\lambda-1}-A^\lambda\leq(\lambda-1)B^\lambda. \tag{3.6}$$

We get,

$$\tilde{w}'(\xi)\leq -m\tilde{b}(\xi)\tilde{\delta}(\xi)+\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\left(\tilde{\delta}'(s)\right)^2}{\tilde{\delta}(s)},\xi\geq\xi_1.$$

Integrating both sides from ξ_1 to ξ, we have

$$\tilde{w}(\xi)\leq\tilde{w}(\xi_1)-\int_{\xi_1}^{\xi}\left(m\tilde{b}(s)\tilde{\delta}(s)+\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\left(\tilde{\delta}'(s)\right)^2}{\tilde{\delta}(s)}\right)ds.$$

Letting $\xi\to\infty$, we get $\lim_{\xi\to\infty}\tilde{w}(\xi)\leq-\infty$, which contradicts the hypothesis (3.1). The proof is complete.

Corollary: 3.1 Let the hypothesis of Theorem 3.1 hold, (3.1) can be replaced by

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$$\limsup_{\xi\to\infty}\int_{\xi_1}^{\xi} m\,\tilde{b}(s)\tilde{\delta}(s)ds=\infty, \tag{3.7}$$

and

$$\limsup_{\xi\to\infty}\int_{\xi_1}^{\xi}\frac{1}{M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\left(\tilde{\delta}'(s)\right)^2}{\tilde{\delta}(s)}ds<\infty. \tag{3.8}$$

Then every solution of (1.1) is oscillatory.

For the following theorem, we introduce a class of functions P. Let

$$D_0=\{(t,s): t>s\ge t_0\},\ D=\{(t,s): t\ge s\ge t_0\}.$$

The function $H\in C(D,R)$ is said to belong to the class P, if

(T_1) H (t,t) = 0 for $t\ge t_0$, H (t,s) > 0 for (t,s) $\in D_0$.

(T_2) H has a continuous and non-positive partial derivative $\frac{\partial H}{\partial s}(t,s)$ and h(t,s) = $\frac{\partial H}{\partial s}(t,s)+H(t,s)\frac{\tilde{\delta}'(s)}{\tilde{\delta}(s)}$.

Theorem: 3.2 Suppose that $(A_1)-(A_4)$ hold and assume also that there exist H $\in P$ such that

$$\limsup_{\xi\to\infty}\frac{1}{\mathrm{H}(\xi,\xi_0)}\int_{\xi_0}^{\xi}\left(\tilde{\delta}(s)\left(m\tilde{b}(s)\mathrm{H}(\xi,s)-\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\mathrm{h}^2(\xi,s)}{\mathrm{H}(\xi,s)}\right)\right)ds=\infty. \tag{3.9}$$

Then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1, we get (3.5). Multiplying (3.5) by $\mathrm{H}(\xi,s)$ on both sides and an integration from ξ_1 to ξ, for $\xi\ge\xi_1$, using (T_2) and (3.6), we obtain

$$\limsup_{\xi\to\infty}\int_{\xi_1}^{\xi}\left(m\tilde{b}(s)\tilde{\delta}(s)\mathrm{H}(\xi,s)-\frac{\tilde{\delta}(s)}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\mathrm{h}^2(\xi,s)}{\mathrm{H}(\xi,s)}\right)ds\le \mathrm{H}(\xi,\xi_0)\,|\,\tilde{\mathrm{w}}(\xi_0)|.$$

Taking limit supremum as $\xi\to\infty$,

$$\limsup_{\xi\to\infty}\frac{1}{\mathrm{H}(\xi,\xi_0)}\int_{\xi_0}^{\xi}\left(\tilde{\delta}(s)\left(m\tilde{b}(s)\mathrm{H}(\xi,s)-\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\mathrm{h}^2(\xi,s)}{\mathrm{H}(\xi,s)}\right)\right)ds\ \le\int_{\xi_1}^{\xi}m\tilde{b}(s)\tilde{\delta}(s)ds+|\,\tilde{\mathrm{w}}(\xi_0)|<\infty,$$

and the latter inequality contradicts to (3.9).

Corollary: 3.2 Let the conditions of Theorem 3.2 hold, equation (3.9) can be replaced by

$$\limsup_{\xi\to\infty}\frac{1}{\mathrm{H}(\xi,\xi_0)}\int_{\xi_0}^{\xi} m\,\tilde{b}(s)\tilde{\delta}(s)\mathrm{H}(\xi,s)ds=\infty, \tag{3.10}$$

and

$$\limsup_{\xi\to\infty}\frac{1}{\mathrm{H}(\xi,\xi_0)}\int_{\xi_0}^{\xi}\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\frac{\mathrm{h}^2(\xi,s)}{\mathrm{H}(\xi,s)}ds<\infty. \tag{3.11}$$

Then every solution of (1.1) is oscillatory.

Let $\mathrm{H}(\xi,s)=(\xi-s)^{\mathrm{n}-1}$, $(\xi,s)\in$ D for some integer n > 2. Then Theorem 3.2 gives the following result.

Corollary: 3.3 Let the assumptions of Theorem 3.2 hold, equation (3.9) can be written as

$$\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)^{\mathrm{n}-1}}\int_{\xi_0}^{\xi}\left(\tilde{\delta}(s)(\xi-s)^{\mathrm{n}-1}\left(m\tilde{b}(s)-\frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)}\left(\frac{\tilde{\delta}'(s)}{\tilde{\delta}(s)}-\frac{\mathrm{n}-1}{\xi-s}\right)^2\right)\right)\mathrm{ds}=\infty. \tag{3.12}$$

for some integer n > 2. Then every solution of (1.1) is oscillatory.

Let $\mathbf{H}(\boldsymbol{\xi},\boldsymbol{s})=(\mathbf{R}(\boldsymbol{\xi})-\boldsymbol{R}(\boldsymbol{s}))^{\lambda}$, $\boldsymbol{\lambda}$ is a constant, $\mathbf{R}(\boldsymbol{\xi})=\int_{\xi_1}^{\xi}\frac{1}{\tilde{r}(s)}\boldsymbol{ds}$ and $\lim_{\xi\to\infty}\mathbf{R}(\boldsymbol{\xi})=\infty$. Then Theorem 3.2 gives the following result.

Corollary: 3.4 Let the assumptions of Theorem 3.2 hold, equation (3.9) can be replaced by

$$\limsup_{\xi\to\infty} \frac{1}{(R(\xi)-R(\xi_0))^{\lambda}} \int_{\xi_0}^{\xi} \left(\tilde{\delta}(s)(R(\xi)-R(s))^{\lambda} \left(m\tilde{b}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \left(\frac{\tilde{\delta}'(s)}{\tilde{\delta}(s)} - \frac{\lambda}{\tilde{r}(s)(R(\xi)-R(s))} \right)^2 \right) \right) ds = \infty. \quad (3.13)$$

Then every solution of (1.1) is oscillatory.

Let $\mathbf{H}(\xi, s) = (\log(\frac{\xi}{s}))^{n}$, $\xi > s > \xi_1$, n > 1 is an integer. Then Theorem 3.2 gives the following result.

Corollary: 3.5 Let the assumptions of Theorem 3.2 hold, equation (3.9) can be replaced by

$$\limsup_{\xi\to\infty} \frac{1}{(\log(\frac{\xi}{\xi_0}))^{n}} \int_{\xi_0}^{\xi} \left(\tilde{\delta}(s) \left(\log(\frac{\xi}{s})\right)^{n} \left(m\tilde{b}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \left(\frac{\tilde{\delta}'(s)}{\tilde{\delta}(s)} - \frac{n}{s\log(\frac{\xi}{s})} \right)^2 \right) \right) ds = \infty. \quad (3.14)$$

Then every solution of (1.1) is oscillatory.

Let $\mathbf{H}(\xi, s) = (\int_s^{\xi} \frac{du}{\theta(u)})^{n}$, $\xi > s > \xi_0$, n > 1 is an integer, $\theta: [\xi_0, \infty) \to \mathbf{R}_+$ is a continuous function such that $\lim_{\xi\to\infty} \int_{\xi_0}^{\xi} \frac{du}{\theta(u)} = \infty$. Then Theorem 3.2 yields the following result.

Corollary: 3.6 Let the assumptions of Theorem 3.2 hold, equation (3.9) can be replaced by

$$\limsup_{\xi\to\infty} \frac{1}{(\int_s^{\xi} \frac{du}{\theta(u)})^{n}} \int_{\xi_0}^{\xi} \left(\tilde{\delta}(s) (\int_s^{\xi} \frac{du}{\theta(u)})^{n} \left(m\tilde{b}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \left(\frac{\tilde{\delta}'(s)}{\tilde{\delta}(s)} - \frac{n}{\theta(s)(\int_s^{\xi} \frac{du}{\theta(u)})} \right)^2 \right) \right) ds = \infty. \quad (3.15)$$

Then every solution of (1.1) is oscillatory.

IV. EXAMPLES

In this section, we give examples to illustrate our main results.

Example: 4.1. Consider the coupled system of fractional nonlinear differential equations

$$\begin{aligned} D_+^{\frac{1}{2}}(u(t)) &= \frac{1}{\sqrt{2}}\left(\cos t + \sqrt{1-\cos^2 t}\right), \\ D_+^{\frac{1}{2}}(v(t)) &= -\frac{\sin t - \cos t}{2\sqrt{\pi}(\sin t\, C(x) - \cos t\, S(x))} \int_0^t (t-s)^{-\frac{1}{2}} u(s)ds, \ t \ge t_0, \end{aligned} \quad (4.1)$$

Here $\propto = \frac{1}{2}, a(t) = \frac{1}{\sqrt{2}}, b(t) = \frac{\sin t - \cos t}{2\sqrt{\pi}(\sin t\, C(x) - \cos t\, S(x))}$, where C(x) and S(x) are the Fresnel integrals, that is

$$C(x) = \int_0^x \cos\left(\frac{1}{2}\pi t^2\right) dt, \ S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt \quad (4.2)$$

with $|C(x)| \le \pi$, $|S(x)| \le \pi$, $f(v) = v + \sqrt{1-v^2}$ and $g(u) = u$.
It is easy to see that $f'(v) = v'(1 - \frac{v}{\sqrt{1-v^2}}) > \varepsilon = m > 0$.
If we take $\delta(t) = 1$ then $\delta'(t) = 0$.
Consider

$$\limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \left[m\tilde{b}(s)\tilde{\delta}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \frac{(\tilde{\delta}'(s))^2}{\tilde{\delta}(s)} \right] ds$$
$$= \limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \varepsilon \frac{\sin s - \cos s}{2\sqrt{\pi}(\sin s\, C(x) - \cos s\, S(x))} ds$$
$$> \limsup_{\xi\to\infty} \frac{\varepsilon}{2\pi\sqrt{\pi}} \int_{\xi_0}^{\xi} \frac{\sin s - \cos s}{\sin s - \cos s} ds \to \infty \ as\ \xi \to \infty.$$

All the conditions of Theorem 3.1 are satisfied. Hence every solution of (4.1) is oscillatory. Infact $(u(t), v(t)) = (\sin t, \cos t)$ is one such solution of (4.1).

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Example: 4.2. Consider the fractional coupled system of nonlinear differential equations

$$D_+^{\frac{1}{2}}(u(t)) = \frac{1}{\sqrt{2}}\left(\sqrt{1-sin^2 t} - \sin t\right),$$
$$D_+^{\frac{1}{2}}(v(t)) = -\frac{\sin\ t+\cos\ t}{2\sqrt{\pi}(\cos\ t\, C(x) - \sin\ t\, S(x))}\int_0^t (t-s)^{-\frac{1}{2}} u(s)ds,\ t \geq t_0, \tag{4.3}$$

Here $\propto = \frac{1}{2}, a(t) = \frac{1}{\sqrt{2}}, b(t) = \frac{\sin\ t+\cos\ t}{2\sqrt{\pi}(\cos\ t\, C(x) - \sin\ t\, S(x))}$, where C(x) and S(x) are given in (4.2)
with the condition $|C(x)| \leq \pi$, $|S(x)| \leq \pi$, $f(v) = v + \sqrt{1-v^2}$ and $g(u) = u$.
It is clear that $f'(v) = v'(1 - \frac{v}{\sqrt{1-v^2}}) > \varepsilon = m > 0$.
Assume $\delta(t) = 1$ then $\delta'(t) = 0$.
Consider

$$\limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \left[m\tilde{b}(s)\tilde{\delta}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \frac{(\tilde{\delta}'(s))^2}{\tilde{\delta}(s)} \right] ds$$
$$= \limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \varepsilon \frac{\sin\ s + \cos s}{2\sqrt{\pi}(\cos s\, C(x) + \sin s\, S(x))} ds$$
$$> \limsup_{\xi\to\infty} \frac{\varepsilon}{2\pi\sqrt{\pi}} \int_{\xi_0}^{\xi} \frac{\sin\ s + \cos s}{\sin s + \cos s} ds \to \infty\ as\ \xi \to \infty.$$

All the conditions of Theorem 3.1 are satisfied. Hence every solution of (4.3) is oscillatory. Infact, $(u(t), v(t)) = (\cos t, -\sin t)$ is one such solution of (4.3).

Example: 4.3. Consider the coupled system of fractional differential equations

$$D_+^{\frac{1}{2}}(u(t)) = e^t f(v(t)),$$
$$D_+^{\frac{1}{2}}(v(t)) = -\frac{1}{\pi t e^t}\frac{1}{\gamma^*(\frac{1}{2},t)}\int_0^t (t-s)^{-\frac{1}{2}} u(s)ds,\ t \geq t_0, \tag{4.4}$$

Here $\propto = \frac{1}{2}, a(t) = e^t, b(t) = \frac{1}{\pi t e^t}\frac{1}{\gamma^*(\frac{1}{2},t)}, f(v) = v^2, g(u) = u$ and $g'(u) = 1 = M > 0$.
Now, to see that $vf(v) < 0$, $f'(v) = 0 = m$, then A_4 fails to hold.
If we choose $\delta(t) = e^{-t}$ then $\delta'(t) = -e^{-t}$.
Consider

$$\limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \left[m\tilde{b}(s)\tilde{\delta}(s) - \frac{1}{4M\Gamma(1-\alpha)\tilde{a}(s)} \frac{(\tilde{\delta}'(s))^2}{\tilde{\delta}(s)} \right] ds$$
$$= \limsup_{\xi\to\infty} \int_{\xi_0}^{\xi} \frac{1}{4\Gamma\left(\frac{1}{2}\right)} e^{-2s} ds < \infty\ as\ \xi \to \infty.$$

All conditions of Theorem 3.1 are not satisfied, since A_4 violates. Hence there is a nonoscillatory solution, $(u(t), v(t)) = (e^t, -1)$ is such solution of (4.4).

V. CONCLUSION

In this paper, we have established some oscillation results for the fractional order differential system using differential inequality method and Riccati transformation. The results obtained are essentially new, have improved and extended some of the results already prevailing in the existing literature. The main results are illustrated with the suitable examples.

REFERENCES

[1] S. Abbas, M. Benchohra, and G. M. N'Guerekata, "Topics in fractional differential equations," Springer, Newyork, 2012.
[2] D. Bainov and D. Mishev, "Oscillatory theory of operator-differential equations", World scientific, Singapore, 1987.

[3] D. X. Chen, “Oscillation criteria of fractional differential equations”, Adv. Differ. Equ., vol. 33, 2012, pp. 1-10.
[4] V. Daftardar-Gejji, “Fractional calculus theory and applications,” Narosa publishing house Pvt. Ltd., 2014.
[5] K. Diethelm, “The analysis of fractional differential equations,” Springer, Berlin, 2010.
[6] L. H. Erbe, Q. K. Kong, and B. G. Zhang, “Oscillation theory for functional differential equations,” Marcel Dekker, New York, 1995.
[7] I. Gyori and G. Ladas, “Oscillation theory of delay differential equations with applications,” Clarendon press, Oxford, 1991.
[8] Q. Feng and F. Meng, “Oscillation of solutions to nonlinear forced fractional differential equations,” Electron. J. Differ. Eq., vol. 169, 2013, pp.1-10.
[9] Z. Han, Y. Zhao, Y. Sun, and C. Zhang, “Oscillation for a class of fractional differential equations,” Hindawi publishing corporation, 2013, Article ID 390282, pp. 1-6.
[10] R. Hilfer, “Applications of fractional Calculus in physics,” World scientific pub. co., 2000.
[11] H. F. Huo and W. T. Li, “Oscillation criteria for certain two-dimensional differential systems,” International journal of applied mathematics, vol. 6, no. 3, 2001, pp. 253-261.
[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, “Theory and applications of fractional differential equations,” Elsevier science, B.V. Amsterdam, 2006.
[13] I-G. E. Kordonis and Ch. G. Philos, “On the oscillation of nonlinear two-dimensional differential systems,” Proc. Amer. Math. Soc., vol. 126, 1998, pp. 1661-1667.
[14] G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, “Oscillation theory of differential equations with deviating arguments,” Pure Appl. Math., Marcel Dekker, Vol.110, 1987.
[15] W. T. Li and S. S. Cheng, “Limiting behaviors of non-oscillatory solutions of a pair of coupled nonlinear differential equations,” Proc. of the Edinburgh Math. Soc., vol. 43, 2000, pp. 457-473.
[16] W. T. Li, “Classification schemes for positive solutions of nonlinear differential systems, Mathematical and computer modelling, vol. 36, 2002, pp. 411-418.
[17] T. Liu, B. Zheng, and F. Meng, “Oscillation on a class of differential equations of fractional order,” Mathematical problems in engineering, 2013, Article ID 830836, pp. 1-13.
[18] A. Lomtatidze and N. Partsvania, “Oscillation and nonoscillation criteria for two-dimensional systems of first order linear ordinary differential equations,” Georgian Math. J., vol. 6, no. 3, 1999, pp. 285-298.
[19] A. Lomtatidze and J. Sremr, “On oscillation and nonoscillation of two-dimensional linear differential systems,” Georgian Math. J., vol. 20, 2013, pp. 573-600.
[20] I. Podlubny, “Fractional differential equations,” Academic press, San Diego, Calif, USA, 1999.
[21] V. Sadhasivam and J. Kavitha, “Forced oscillation of fractional order partial differential equations with damping and functional arguments,” IJPAM, vol. 106, no. 7, 2016, pp. 89-107.
[22] S. H. Saker, “Oscillation theory of delay differential and difference equations,” VDM Verlag Dr.Muller Aktiengesellschaft and Co, USA, 2010.
[23] S. G. Samko, A. A. Kilbas, and O. I. Mirchev, “Fractional integrals and derivatives, Theory and applications,” Gordon and breach science publishers, Singapore, 1993.
[24] V. E. Tarasov, “Fractional dynamics,” Springer, 2009.
[25] P. Weng and Z. Xu, “On the oscillation of a two-dimensional delay differential system,” International journal of qualitative theory of differential equations and applications, vol. 2, no. 1, 2008, pp. 38-46.
[26] Y. Zhou, “Basic theory of fractional differential equations,” World scientific, Singapore, 2014.