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Euler Numbers of Binary Images in Grid Cell Representation

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ABSTRACT: An alternative representation of binary image data is described, The corresponding Euler number formulas are derived.

I. BASICS

As usual, $\mathbf{Z}^2$ denotes the integer grid in the plane. We define a binary digital image as a function $P:\mathbf{Z}^2 \to \{0,1\}$. A coordinate system in $\mathbf{Z}^2$ is chosen such that the first axis points downward (the *row* axis) and the second axis points to the right (the *column* axis). An element $(i, j) \in \mathbf{Z}^2$ can be regarded as a point placed at row i and column j, or as a square placed with its centre at coordinates (i, j); such an element is usually called a *pixel*. We call this the pixel representation of the image P. Alternatively, the element $(i, j) \in \mathbf{Z}^2$ can be regarded as a point placed at coordinates $(i+1/2, j+1/2)$, or as a square placed with its upper-left corner at coordinates (i, j). If $P(i, j)=0$, the pixel (i, j) is called a *background* point; otherwise, if $P(i, j)=1$, the pixel (i, j) is called a *foreground* point. We will assume in this paper that background pixels (0-pixels) are black and foreground pixels (1-pixels) are white. We also assume the number of 1-pixels to be finite; we can therefore restrict each image to a digital rectangle.

In this paper, we use the *grid cell representation* of the image P, where the faces (cells) are squares whose boundaries lie on integer coordinates. Nearby pixels on a row or column are always seen as connected (each pixel has at least 4 neighbours). For nearby pixels in a diagonal relationship we have a few choices; such a pair of pixels can be defined as connected or as disconnected. In the N8 (eight neighbours) connectivity, the pair of pixels is considered connected, both in the main diagonal direction and in the secondary diagonal direction. In the N6m (six neighbours, main diagonal) connectivity, the pair of pixels is considered connected only in the main diagonal direction. In the N6s (six neighbours, secondary diagonal) connectivity, the pair of pixels is considered connected only in the secondary diagonal direction. In the N4 (four neighbours) connectivity, the pair of pixels is never considered connected.

We denote this graph by $grid(P)$. The faces of this graph are single pixels, represented as $[1]$; its vertical edges are horizontal pairs $[x \ x]$, where not both values are 0; its horizontal edges are vertical pairs $\begin{bmatrix} x \\ x \end{bmatrix}$, where not both values are 0; and its vertices are quadruples of pixels represented as $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$, where not all four values are 0. In the cases of diagonally disconnected pairs of pixels, the vertex is counted twice; however, if the pixels are diagonally connected, the two copies of the vertex are connected by an edge and the vertex is counted only once (see the drawings below).

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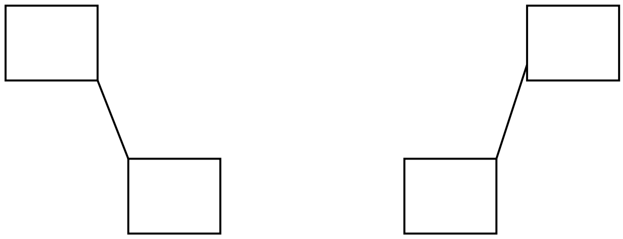
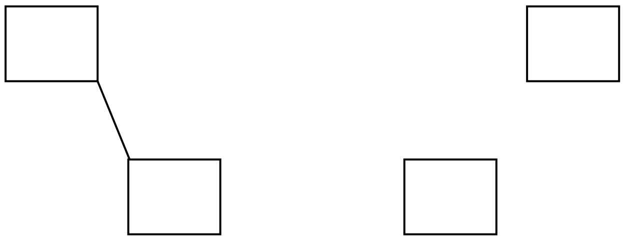
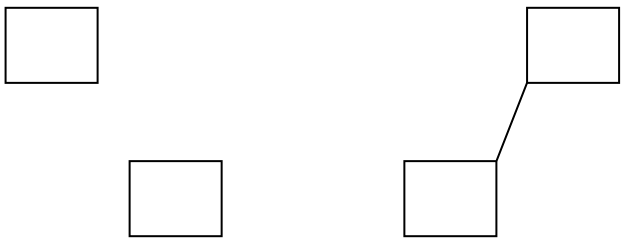
N8



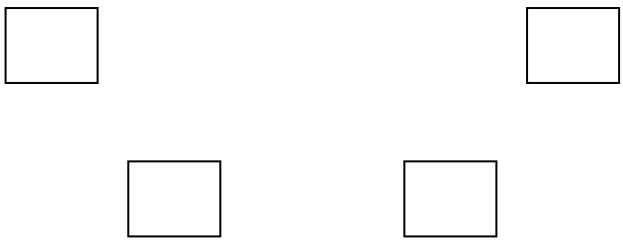
N6m



N6s



N4



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II. DEFINITION OF THE EULER NUMBERS

For any binary image P , we define:

- the number of faces in the graph $grid(P)$: $f(P)=\#([1])$;
- the number of vertical edges in the graph $grid(P)$: $ve(P)=\#(P;[0\ \ 1])+\#(P;[1\ \ 0])+\#(P;[1\ \ 1])$;
- the number of horizontal edges in the graph $grid(P)$: $he(P)=\#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)$.

According to the connectivity chosen, we also define the number of vertices in the graph $grid(P)$:

- for N8:

$$\begin{aligned} v^{(8)}(P) = {} & \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) \\ & +\#\left(P;\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&0\end{bmatrix}\right) \\ & +\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right) \\ & +\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) \end{aligned}$$

- for N6m: $v^{(6m)}(P)=v^{(8)}(P)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$
- for N6s: $v^{(6s)}(P)=v^{(8)}(P)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$
- for N4: $v^{(4)}(P)=v^{(8)}(P)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$

To be more precise: each vertical edge in the grid is defined by a horizontal bit-pair (the vertical common side of the two pixels); each horizontal edge in the grid is defined by a vertical bit-pair (the horizontal common side of the two pixels); and each vertex in the grid is defined by a bit-quad (the four pixels around this vertex).

The Euler numbers for the four connectivity types are defined as follows:

$$\begin{aligned} \chi^{(8)}(P) &= f(P)-ve(P)-he(P)+v^{(8)}(P) \\ \chi^{(6m)}(P) &= f(P)-ve(P)-he(P)+v^{(6m)}(P) \\ \chi^{(6s)}(P) &= f(P)-ve(P)-he(P)+v^{(6s)}(P) \\ \chi^{(4)}(P) &= f(P)-ve(P)-he(P)+v^{(4)}(P) \end{aligned}$$

For the 4-connectivity and 8-connectivity Euler numbers, see the references [1], [3, p.349], [4, pp.245-247]; for the two 6-connectivity Euler numbers see the references [1], [2, pp.76-77 and p.85].

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III. THE BIT-QUAD REPRESENTATION

We can represent each of the smaller digital patterns whose counts appear in the Euler number definitions, as a bit-quad (2×2 digital square) containing it in its left-upper corner; and thus we get

$$
\begin{aligned}
f(P) &= \#\left(P;[1]\right) = \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\\
ve(P) &= \#\left(P;[0\ 1]\right)+\#\left(P;[1\ 0]\right)+\#\left(P;[1\ 1]\right)\\
&= \#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\\
he(P) &= \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)\\
&= \#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\\
v(P) &= \#\left(P;\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)\\
&\quad +\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)
\end{aligned}
$$

These lead us to the following identities

$$
\begin{aligned}
\chi^{(8)}(P) &= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\
\chi^{(6m)}(P) &= \chi^{(8)}(P)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=\#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\
\chi^{(6s)}(P) &= \chi^{(8)}(P)+\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\
&= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)\\
\chi^{(4)}(P) &= \chi^{(8)}(P)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)\\
&= \#\left(P;\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)-\#\left(P;\begin{bmatrix}0&1\\1&1\end{bmatrix}\right)+\#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)
\end{aligned}
$$

and to their compact variants (*x* denotes either of the values 0 or 1)

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$$\chi^{(8)}(P) = \#\left(P;\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & X \end{bmatrix}\right)$$
$$\chi^{(6m)}(P) = \#\left(P;\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$\chi^{(6s)}(P) = \#\left(P;\begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & X \end{bmatrix}\right)$$
$$\chi^{(4)}(P) = \#\left(P;\begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

When computing an Euler number for some binary digital image, we have to restrict the image domain to some rectangle containing all white pixels; however, we have to make sure for our images that one of the rows (first or last) is completely black and that one of the columns (first or last) is completely black.

For each of these eight identities, we have three more variants, according to the direction of the corner into which the small patterns are placed in the bit-quads; (see [5] for the complete list).

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