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Modification of Weerakoon-Fernando's Method with Fourth-Order of Convergence for Solving Nonlinear Equation

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ABSTRACT: Weerakoon-Fernando's method is an iterative method to solve a nonlinear equation with third order of convergence that involve three evaluation of functions. In this paper, the author developed the method by using centroidal means and second order Taylor's polynomial, then its second derivative is reduced by using approximation of sum of forward and backward difference. Based on the convergence analysis, we show that the new iterative method converges quartically and involves three evaluations of function with efficiency index equal to $4^{1/3}\approx 1.5874$. Numerical simulation present that the proposed method is comparable to other methods discussed and gives better results.

KEYWORDS: centroidal means, nonlinear equation, order of convergence, Taylor's polynomial, Weerakoon-Fernando's method.

I. INTRODUCTION

Nonlinear equation is mathematical representation from various phenomenon of scientific and engineering field. The problem of the nonlinear equation is how to solve the equation by using analytical technique. Therefore, many nonlinear equations are not able to solve analytical technique, then numerical solving is an alternative solution that use looping calculated. It is known widely as iterative method. In this paper, we consider iterative method to solve a simple root from a nonlinear equation in form

$$f(x) = 0, \tag{1}$$

where $f : D\subset\Re\to\Re$ for open interval I is a scalar function.

The classical iterative method that known widely as basic approximation for solving nonlinear equation is written as follow

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

The method of (2) converges quadratically with efficiency index equal to $2^{1/2}\approx 1.4142$.

In order to improve the local of order of convergence in (2), many technical approximations have been proposed to construc a new iterative method which converges cubically, such as quadratic polynomial [1, 13], Adomian decomposition method [3, 4], circle of curvature [7], and modification of Newton's method [2, 4, 6, 8, 11,12, 14, 16, 19]. The last technical approximation appear a iterative method that is known widely as two point iterative method.

One of the two point iterative method is a Weerakoon-Fernando's method [19] written in

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \tag{3}$$

Iterative method (3) is a third-order of convergence involving three functional evaluations with efficiency index equal to $3^{1/3}\approx 1.4422$. Based on Traub [17], the order of convergence of an iterative method will be optimal if the efficiency index of the iterative method satisfies $2^{n/(n+1)}$ where n is number of point of the iterative method. For two point iterative

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method with three functional evaluations, order of convergence of the method will be optimal if its order of convegence is four.

In this paper, we developed the Weerakoon-Fernando's method using second order Taylor series expansion and reduced its second derivative by using an equality of Newton-Steffensen method with one real parameter [15] and Weerakoon-Fernando's method [19]. Furthermore, a numerical simulation is given to compare the performance of the proposed method and then compare its performance with other methods.

II. THE SCHEMATIC OF WEERAKOON-FERNANDO'S ITERATIVE METHOD

Weerakoon-Fernando's iterative method has been developed by using Newton's theorem and trapezoidal rule, which is written as following, respectively

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{4}$$

and

$$\int_{x_n}^{x} f'(t)dt = (x - x_n)\left(\frac{f'(x) + f'(x_n)}{2}\right) \tag{5}$$

Substitute (5) into (4), we get

$$f(x) = f(x_n) + (x - x_n)\left(\frac{f'(x) + f'(x_n)}{2}\right) \tag{6}$$

If we take $x = x_{n+1}$, then (6) can be written as

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)\left(\frac{f'(x_{n+1}) + f'(x_n)}{2}\right) \tag{7}$$

Suppose, x_{n+1} is a approximation root of nonlinear equation at $(n+1)$th iteration which converges closely to α,so that$f(x_{n+1}) \approx 0$. Therefore, we can wrtie (7) as form

$$f(x_n) + (x_{n+1} - x_n)\left(\frac{f'(x_{n+1}) + f'(x_n)}{2}\right) = 0 \tag{8}$$

or

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_{n+1}^*) + f'(x_n)} \tag{9}$$

where x_{n+1}^* is Newton's method defined in (2).

Equation (9) is third-order iterative method involving three evaluation functions with efficiency index equal to $3^{1/3} \approx 1.4422$. The iterative method is widely known as Weerakoon-Fernando's method which is commonly written as (3).

III. THE PROPOSED METHOD AND CONVERGENCE ANALYSIS

To construct a new iterative method, we consider Weerakoon-Fernando's method in (3) as arithmetic means form

$$x_{n+1} = x_n - \frac{f(x_n)}{\left(\frac{f'(x_n) + f'(y_n)}{2}\right)}, \tag{10}$$

wherey_n is given by (2).

Furthermore, suppose $a = f'(x_n)$ and $b = f'(y_n)$, then by using centroidal means, the Equation of (10) can be written as

$$x_{n+1} = x_n - \frac{3f(x_n)\left(f'(x_n) + f'(y_n)\right)}{2\left(f'(x_n)^2 + f'(x_n)f'(y_n) + f'(y_n)^2\right)}. \tag{11}$$

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The equation of (11) is variant of Weerakoon-Fernando's method with third-order of convergence. Therefore, to improve the method, we consider a second order Taylor series expansion about x_n as form

$$f(x_n) \approx f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(x_n)(x - x_n). \tag{12}$$

Suppose x_{n+1} is a root aproximation for nonlinear equation that closely to an its exact root α, then $(x_{n+1}) \approx 0$. Therefore, the second order Taylor series expansion at$x = x_{n+1}$in (12) can be formed as

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} - \frac{(x^*_{n+1} - x_n)^2 f''(x_n)}{2 f'(x_n)}, \tag{13}$$

where x^*_{n+1} is given by equation of (11). Then, the equation (11) substitutes into (13), we can write (13) as follow

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} - \frac{9\left(f'(x_n) + f'(y_n)\right)^2 f(x_n)^2 f''(x_n)}{8\left(f'(x_n)^2 + f'(x_n) f'(y_n) + f'(y_n)^2\right)^2 f'(x_n)}. \tag{14}$$

Equation (14) is a two-point iterative method with four evaluations of functions and involves second dervative of $f(x_n)$. To avoid the second derivative, we reduce it by using sum of forward and backward difference as form

$$f''(x_n) \approx \frac{f'(x_n)^2 - f'(x_n) f'(y_n)}{f(x_n)}. \tag{15}$$

Substitute (15) into (14), we get a iterative method without second derivative

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} - \frac{9 f(x_n)\left(f'(x_n) - f'(y_n)\right)\left(f'(x_n) + f'(y_n)\right)^2}{8\left(f'(x_n)^2 + f'(x_n) f'(y_n) + f'(y_n)\right)^2}. \tag{16}$$

Furthermore, $f'(y_n)$ in (16) is approximated by using equality of Newton-Steffensen method with one real parameter [7] and variant of Newton method [8] as form

$$f'(y_n) \approx \frac{f'(x_n)\left(f(x_n) - (2-\theta) f(y_n)\right)}{f(x_n) + \theta f(y_n)}. \tag{17}$$

Subtitute $f'(y_n)$ in (16) by using (17) then we will get a new iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{18}$$

$$x_{n+1} = x_n - \left(1 + \frac{9 f(y_n)\left(f(x_n)^2 + 2(\theta-1) f(x_n) f(y_n) + (\theta-1)^2 f(y_n)^2\right)\left(f(x_n) + \theta f(y_n)\right)}{\left(3 f(x_n)^2 + 6(\theta-1) f(x_n) f(y_n) + \left(3\theta^2 - 6\theta + 4\right) f(y_n)^2\right)^2}\right) \frac{f(x_n)}{f'(x_n)}. \tag{19}$$

Equation (18) – (19) is a two point iterative method that involve two evaluation of functions f and one its derivative. Based on Traub [17], the propose method will be a optimal iterative method if its order of convergence is four.

Theorem 2. Let $f: D \subset \Re \to \Re$ be a differentiable function in open interval D. Then assume that is a simple root of $f(x) = 0$. Suppose x_0 is a given value that is sufficiently close toα, then iterative method (18) – (19) has fourth-order of convergence for $\theta = 0$ with error

$$e_{n+1} = \left(-c_2 c_3 + \frac{8}{3} c_2^3\right) e_n^4 + O(e_n^5), \tag{20}$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}, k = 2,3,4,5,\ldots$.

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Proof. Let α is an exact root of $f(x)$, then $f(\alpha)=0$. Furthermore, assume $f'(x)\neq 0$ and $x_n = e_n + \alpha$, then Taylor series expansion for $f(x_n)$ and $f'(x_n)$ about α, we have

$$f(x_n)=f'(\alpha)\left(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+O(e_n^5)\right), \tag{21}$$

and

$$f'(x_n)=f'(\alpha)\left(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+\ldots+O(e_n^5)\right). \tag{22}$$

Using (21) and (22), we have

$$\frac{f(x_n)}{f'(x_n)}=e_n-c_2e_n^2+(2c_2^2-2c_3)e_n^3+\ldots+O(e_n^5),$$

and hence,

$$y_n=\alpha+c_2e_n^2+2(-c_2^2+c_3)e_n^3+\ldots+O(e_n^5). \tag{23}$$

Again, use expansion series Taylor for $f(y_n)$ about α, we have

$$f(y_n)=f'(\alpha)(c_2e_n^2+(2c_3-2c_2^2)e_n^3+\ldots+O(e_n^5)). \tag{24}$$

Based on (21), (22) and (24), we have

$$f(x_n)^2=f'(\alpha)^2\left(e_n^2+2c_2e_n^3+(2c_3+c_2^2)e_n^4+O(e_n^9)\right), \tag{25}$$

and

$$f(y_n)^2=f'(\alpha)^2\left(c_2^2e_n^4+(2c_3-2c_2^2)e_n^5+O(e_n^6)\right). \tag{26}$$

Furthermore, use (21), (22), (24), (25) and (26), we found

$$f(x_n)+\theta f(y_n)=f'(\alpha)\left(e_n+(1+\theta)c_2e_n^2+((1+2\theta)c_3-2\theta {c_2}^2)e_n^3+(5\theta c_2^3-7\theta c_2c_3+(1+3\theta)c_4)e_n^4+O(e_n^5)\right), \tag{27}$$

$$f(x_n)^2+2(\theta-1)f(x_n)f(y_n)+(\theta-1)^2f(y_n)^2=f'(\alpha)^2\left(e_n^2+2\theta c_2e_n^3+\left((\theta-2)^2c_2^2+2(2\theta-1)c_3\right)e_n^4+O(e_n^5)\right), \tag{28}$$

$$3f(x_n)^2+6(\theta-1)f(x_n)f(y_n)+(3\theta^2-6\theta+4)f(y_n)^2=f'(\alpha)^2\left(3e_n^2+6\theta c_2e_n^3+\left((3\theta^2-12\theta+13)c_2^2+6(2\theta-1)c_3\right)e_n^4 +O(e_n^5)\right), \tag{29}$$

and

$$f(x_n)f(y_n)=f'(\alpha)^2(c_2e_n^3+(2c_3-{c_2}^2)e_n^4+O(e_n^5)). \tag{30}$$

Use (25), (26), (27), (28), (29), (30) and $x_{n+1} = e_{n+1} + \alpha$, we will get

$$e_{n+1}=\theta c_2^3e_n^3+\left((4\theta-1)c_2c_3-\frac{1}{3}\left(3\theta^2+9\theta-8\right)c_2^3\right)e_n^4+O(e_n^5). \tag{31}$$

Equation (31) give an information that the convergence order of (18) – (19) will increase if we take $\theta = 0$. So, by subtituting again $\theta = 0$ into (31), we get

$$e_{n+1}=\left(-c_2c_3+\frac{8}{3}c_2^3\right)e_n^4+O(e_n^5). \qquad \blacksquare \tag{32}$$

The proposed method (18) – (19) requires three evaluations of function per iteration. Based on definition of efficiency index, i.e *I.E* $= p^{1/m}$, where *p* is the order of the iterative method and *m* is the number of functional evaluations per iteration, the propose method in (18) – (19) have the effeciency index equals to $4^{1/3} \approx 1.5874$, which is better than the Newton's method $2^{1/2} \approx 1.1442$ [17], Halley's method $3^{1/3} \approx 1.4422$ [13], Weerakoon-Fernando's method $3^{1/3} \approx 1.4422$ [19].

Theproposed iterative method in (18) – (19) with one real parameter θ will appear one fourth-order and some third-order iterative methods.
For $\theta = 0$, we get fourth-order iterative method that is given by

$$x_{n+1}=x_n-\left(1+\frac{9f(x_n)f(y_n)\left(f(x_n)-f(y_n)\right)^2}{\left(3f(x_n)\left(f(x_n)-2f(y_n)\right)+4f(y_n)^2\right)^2}\right)\frac{f(x_n)}{f'(x_n)}. \tag{33}$$

For $\theta = 1$, the equation (19) presented a third-order iterative method

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$$x_{n+1} = x_n - \left(1 + \frac{9\left(f(x_n)+f(y_n)\right)f(x_n)^2}{\left(3f(x_n)^2+f(y_n)^2\right)^2}\right)\frac{f(x_n)}{f'(x_n)}. \quad (34)$$

For θ= 2, the equation(19) presented a third-order iterative method

$$x_{n+1} = x_n - \left(1 + \frac{9f(y_n)\left(f(x_n)+f(y_n)\right)^2\left(f(x_n)+2f(y_n)\right)}{\left(3f(x_n)\left(f(x_n)+2f(y_n)\right)+4f(y_n)^2\right)^2}\right)\frac{f(x_n)}{f'(x_n)}. \quad (35)$$

IV. NUMERICAL SIMULATION

In this section, we presented numerical simulation to show the performance of the proposed method. The indicators of the performance involved the number of iteration, computational of order of convergence (COC), absolute value of function, value of absolute error. The result of numerical simulation of the proposed method both of $\theta \neq 0$ (MT1) and $\theta = 0$ (MT2) are compared with Newton's method (MN) [17], Halley's method (MH) [13], and Weerakoon-Fernando's method (MWF) [19].

In this numerical simulation, all computing are performed by using MAPLE 13.0 with 850 digits floating point arithmetic's and the following stopping criterion for computer program

$$|x_{n+1} - x_n| \le \varepsilon. \quad (36)$$

In addition, when the stopping criterion is satisfied, then x_{n+1} is taken as the exact root α. Furthermore, COC may be approximated as

$$\rho \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \quad (37)$$

We used the following eight test function while its exact roots displayed the computed approximate zeros α round up to 15th decimal places

$f_1(x) = xe^{-x} - 0{,}1$, $\alpha \approx 0.111832559158962$,

$f_2(x) = e^x - 4x^2$, $\alpha \approx 4.306584728220692$,

$f_3(x) = \cos(x) - x$, $\alpha \approx 0.739085133215160$,

$f_4(x) = (x-1)^3 - 1$, $\alpha = 2.000000000000000$,

$f_5(x) = x^3 + 4x^2 - 10$, $\alpha \approx 1.365230034140968$,

$f_6(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$, $\alpha = -1.000000000000000$,

$f_7(x) = \sin^2(x) - x^2 + 1$, $\alpha \approx 1.404491648215341$,

$f_8(x) = \sqrt{x} - x$, $\alpha \approx 1.000000000000000$.

Table 1 shows the number of iteration (IT) requiring such thatsatifies (36) where $\varepsilon = 10^{-95}$ and the computational order of convergence (COC) in the parentheses by using formulas (37). In the Table 1, first colmun shows some test functions denoted by f, second column shows initial value denoted by x_0, and third column to seventh column shows number of iteration and COC for five compared iterative methods.

Table 1. The number of iteration and*COC*for several iterative methods with $\varepsilon = 10^{-95}$

$f(x)$	x_0	MN	MH	MWF	MT1 ($\theta \neq 0$)	MT2 $\theta = 0$
$f_1(x)$	–0.2	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	0.3	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_2(x)$	4.0	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	4.5	7(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_3(x)$	–0.1	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	5(4.0000)

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	1.5	7(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_4(x)$	1.8	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	3.0	9(2.0000)	6(3.0000)	6(3.0000)	6(3.0000)	5(4.0000)
$f_5(x)$	1.0	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	2.0	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_6(x)$	–1.5	7(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	0.0	7(2.0000)	6(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_7(x)$	1.2	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
	2.0	8(2.0000)	5(3.0000)	5(3.0000)	5(3.0000)	4(4.0000)
$f_8(x)$	0.5	8(2.0000)	5(3.0000)	4(3.0000)	5(3.0000)	4(4.0000)
	1.5	7(2.0000)	5(3.0000)	4(3.0000)	5(3.0000)	4(4.0000)

Based on the results in the Table 1, we can see that order of convergence of the proposed methods is three for $\theta \neq 0$ and four for $\theta = 0$.From computational order of convergence (COC) of each iterative method, MT2 has a convergency higher order than others. Therefore, the MT2 move faster in reaching the root of nonlinear equation. Beside, Table 1 confirms the order of convergence of the proposed method theoretically.

The accuracy of the proposed method and several methods by using the same total number of functional evaluations(TNFE) as comparison are presented at Table 2.

Table2. Value of absolute error for TNFE = 12

$f(x)$	x_0	MN	MH	MWF	MT1 ($\theta \neq 0$)	MT2 ($\theta = 0$)
$f_1(x)$	–0.2	3.8845 (e–36)	3.4951 (e–55)	2.0386(e–42)	1.2313(e–43)	7.8657(e–120)
	0.3	1.3518 (e–42)	4.4263 (e–66)	1.2806 (e–49)	2.3183 (e–85)	4.9837 (e–136)
$f_2(x)$	4.0	1.2647 (e–34)	5.3111 (e–55)	9.7099(e–40)	5.5171(e–48)	1.9424(e–103)
	4.5	8.0333 (e–54)	1.3204 (e–77)	1.0794 (e–64)	2.0699 (e–66)	2.3441 (e–187)
$f_3(x)$	–0.1	1.1593 (e–37)	1.6190 (e–43)	4.5010(e–57)	1.0514(e–40)	7.0989(e–091)
	1.5	2.2471 (e–64)	6.8693 (e–52)	3.2365 (e–77)	6.2022 (e–80)	7.6705 (e–195)
$f_4(x)$	1.8	9.5535 (e–42)	5.7624 (e–61)	1.3366(e–49)	1.9849(e–85)	1.4574(e–129)
	3.0	1.5483 (e–16)	2.1303 (e–24)	2.9220 (e–19)	4.2504 (e–19)	2.4574 (e–048)
$f_5(x)$	1.0	2.4116 (e–44)	1.3534 (e–61)	9.0984(e–54)	2.0976(e–83)	3.5405(e–139)
	2.0	7.4858 (e–39)	2.8220 (e–53)	2.6870 (e–47)	2.3802 (e–47)	6.2011 (e–129)
$f_6(x)$	–1.5	9.5649 (e–67)	2.5437 (e–44)	9.2574(e–54)	7.4619(e–88)	5.4987(e–179)
	0.0	3.2102 (e–66)	1.0653 (e–26)	1.4935 (e–36)	1.9411 (e–73)	1.8351 (e–155)
$f_7(x)$	1.2	8.4046 (e–48)	6.2549 (e–65)	1.3027(e–58)	5.1674(e–76)	2.3418(e–151)
	2.0	9.1131 (e–33)	3.4723 (e–39)	6.0204 (e–42)	9.9833 (e–40)	2.8080 (e–103)
$f_8(x)$	0.5	3.0985 (e–43)	5.9334 (e–34)	1.0728(e–182)	5.4699(e–50)	1.6467(e–104)
	1.5	2.1299 (e–66)	4.4256 (e–66)	1.5490 (e–259)	2.9260 (e–83)	8.0342 (e–215)

Table 2 presented that all of comparison methods converge to the root of nonlinear equation for all test functions. The accuracy of the proposed method (18) – (19) for $\theta = 0$ is better than other iterative methods.

V.CONCLUSION

This study discovers a new variant of Weerakoon-Fernando's method as an alternative iterative method to find the roots of the nonlinear equation that require evaluation of two function and one first derivative per iteration with efficiency index equal to $4^{1/3} \approx 1.5874$. Numerical simulation present that the proposed method is better than Weerakoon-Fernando's method and other discussed methods.

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