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Checking the stability of the finite difference schemes for symmetric hyperbolic systems using Fourier transitions

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ABSTRACT. In this article we investigated the stability of the finite difference schemes using Fourier transitions for symmetric hyperbolic systems. Eigen values of symmetric matrices are used to check the stability [1]. In some scientific articles, eigenvalues of application matrices for finite difference schemes are their stability is determined by their unity in the circle. For hyperbolic systems, there are several finite difference schemes that have been studied for their stability.

KEYWORDS: Finite difference scheme, stability conditions, symmetric hyperbolic system.

I.INTRODUCTION

The Neyman method, which analyzes the stability of the finite difference schemes, is the most widely used method. It is also possible to verify that the circuits are stabilized by applying the Fourier transitions [3]. We can write symmetric hyperbolic systems as follows

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{n} A_k \frac{\partial u}{\partial x_k} \qquad (1)$$

here A_k are real constant $N \times N$ dimensional symmetric matrices, F are optional $N \times N$ size matrix. (1) as a result of the approximation of the system, we have the following scheme:

$$u^{n+1} = L(A_k, T)u^n \qquad (2)$$

here $L-$ is the multiplication that depends on the A_k matrix and the T slider operator.

II. APPLY FOURIER TRANSFORM FOR FINITE DIFFERENCE SCHEMES

- Detected for optional $u \in R$

$$\hat{u}(\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\varphi x} u(x)dx, \qquad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\varphi x} \hat{u}(\varphi)d\varphi$$

- According to the mesh spacing h $v = (\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3 \ldots)$ for mesh function (here $\zeta \in [-\pi/h; \pi/h]$)

$$\hat{v}(\zeta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\zeta} v_m h, \qquad v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\zeta} \hat{v}(\zeta)d\zeta$$

$$u^n = \phi^n e^{i\sum_{k=-N}^{N} k\Delta x_k}$$

By using Fourier transform (2), a finite difference scheme

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$$\phi^{n+1} = D(A_k, \gamma_k)\phi^n \qquad (3).$$

Here $D-$ application matrices depend on the Fourier variables γ_k and A_k matrices. Else $\phi_{-1}^n = \phi_{N-1}^n$, $\phi_N^n = \phi_0^n$ boundary conditions for the (3) finite difference scheme then $D-$ application matrices

$$D = \begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_{n-1} \\ d_{n-1} & d_0 & d_1 & \ddots & d_2 \\ d_{n-2} & \ddots & \ddots & \ddots & d_2 \\ \vdots & \ddots & d_{n-1} & d_0 & d_1 \\ d_1 & d_2 & d_3 & \cdots & d_0 \end{pmatrix}$$

is the circular matrix. $D-$circulant matrix has real eigenvalues numbers and it has a diagonalization feature. Thus, the $D-$circulant matrix can be written as $D = P\Lambda P^{-1}$, where $P_{m,n} = \frac{1}{\sqrt{N}} e^{j2\pi m\frac{n}{N}}$ is the discrete Fourier transform matrix, and $\Lambda-$ diagonal matrix is the eigenvalues numbers of $D-$circulant matrix

$$\Lambda = \begin{pmatrix} D(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & D(\lambda_2) & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & D(\lambda_{n-1}) & 0 \\ 0 & 0 & 0 & \cdots & D(\lambda_n) \end{pmatrix}$$

$$P^{-1}\phi^{n+1} = P^{-1}D(A_k, \gamma_k)\phi^n$$

$$P^{-1}EPP^{-1}\phi^{n+1} = P^{-1}D(A_k, \gamma_k)PP^{-1}\phi^n \qquad (4).$$

If $\hat{\phi}^n = P^{-1}\phi^n$ is switched in the (4) finite difference scheme

$$\Lambda_E \hat{\phi}^{n+1} = \Lambda_D \hat{\phi}^n \qquad (5)$$

we will have a finite difference scheme. Here Λ_E, Λ_D are diagonal matrices, composed from the eigenvalues of $E-$ unity and $D-$ application matrices.

III. STUDYING THE CONDITION STABILITY OF THE FINITE DIFFERENCE SCHEMES

Remark. Λ is a $N \times N$ matrix with N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ obtained as solution of the polynomial

$$\det\left|\Lambda - E \cdot I\right| = 0$$

(here $E-$unity matrix, $I = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T$) its spectral radius is defined by the modulus of the largest eigenvalue

$$\rho(\Lambda) = \underset{i=1,N}{\mathrm{Max}}\left|\lambda_i\right|.$$

The Von Neumann necessary condition for stability can be stated as the condition that the spectral radius of the amplification matrix satisfies {5 (Richtmyer and Morton, 1967)}

$$\rho(\Lambda) \le 1 + o(\Delta t)$$

for finite Δt and for all values of ϕ, in the range $(-\pi, \pi)$. This condition is less severe than the previous one, which corresponds to a condition

$$\rho(\Lambda) \le 1. \qquad (6)$$

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According to the above conditions, if diagonal elements of Λ_E , Λ_D diagonal matrices are respectively $\lambda_{E,k}$, $\lambda_{D,k}$, then (2) a finite difference scheme

$$\max\left|\frac{\lambda_{D,k}}{\lambda_{E,k}}\right| \leq 1$$

is stable according to the conditions.

IV. EXPERIMENTAL RESULT

If $n=1$ then (1) system is

$$\frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x} \tag{7}$$

We will use a mesh of equally spaced points (x_j, t_n) where $x_{m+1} - x_m = h$, $t_{n+1} - t_n = \tau$ and $0 \leq m \leq M$, $0 \leq n \leq N$, $\lambda = \frac{\tau}{h}$ (courant number). We use the notation $u_m^n = u(x_m, t_n)$.

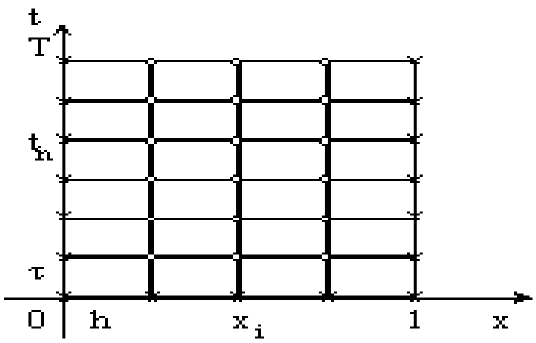

Fig1: Grid according to parameters

Then write down the finite difference schemes for (7) system using the following template.

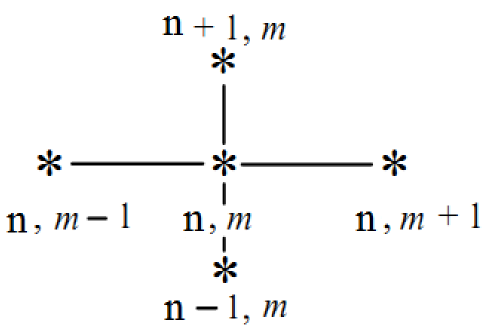

Fig2: Template appearance

Table1. Finite difference schemes for system (7)

№	The name of the finite difference scheme	Expression of the finite difference scheme
1	forward time forward space	$u_m^{n+1} = u_m^n + A\lambda\left(u_m^n - u_{m+1}^n\right)$
2	forward time backward space	$u_m^{n+1} = u_m^n + A\lambda\left(u_m^n - u_{m-1}^n\right)$

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3	forward time centered space	$u_m^{n+1} = u_m^n + A\lambda \frac{u_{m+1}^n - u_{m-1}^n}{2}$
4	backward time centered space	$u_m^n - u_m^{n-1} = A\lambda \frac{u_{m+1}^n - u_{m-1}^n}{2}$
5	Lax-Wendroff	$u_m^{n+1} = u_m^n - A\frac{\lambda}{2}\left(u_{m+1}^n - u_{m-1}^n\right) + + A^2\frac{\lambda^2}{2}\left(u_{m-1}^n - 2u_m^n + u_{m-1}^n\right)$
6	Lax-Friedrichs	$u_m^{n+1} = \left(\frac{1}{2} + A\frac{\lambda}{2}\right)u_{m+1}^n + \left(\frac{1}{2} - A\frac{\lambda}{2}\right)u_{m-1}^n$
7	Crank-Nicolson	$u_m^{n+1} = u_m^n + A\lambda \frac{u_{m+1}^{n+1} - u_{m-1}^{n+1} + u_{m+1}^n - u_{m-1}^n}{4}$
8	leap-frog	$\frac{u_m^{n+1} - u_m^{n-1}}{2} = A\lambda \frac{u_{m-1}^n - u_{m+1}^n}{2}$

In the above Lax-Friedrichs schemes, we can perform fourier transforms

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\zeta} \hat{v}(\zeta) d\zeta ,$$

as a result

$$u_m^{n+1} = \left(\frac{1}{2} + A\frac{\lambda}{2}\right)u_{m+1}^n + \left(\frac{1}{2} - A\frac{\lambda}{2}\right)u_{m-1}^n$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\zeta} \hat{v}^{n+1}(\zeta) d\zeta = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{i(m+1)h\zeta} \left(\frac{1}{2} + A\frac{\lambda}{2}\right)\hat{v}^n(\zeta) d\zeta +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{i(m-1)h\zeta} \left(\frac{1}{2} - A\frac{\lambda}{2}\right)\hat{v}^n(\zeta) d\zeta .$$

We can simplify

$$\hat{v}^{n+1}(\zeta) = \left(\left(\frac{1}{2} + A\frac{\lambda}{2}\right)e^{ih\zeta} + \left(\frac{1}{2} - A\frac{\lambda}{2}\right)e^{-ih\zeta}\right)\hat{v}^n(\zeta) = G(h\zeta)\hat{v}^n(\zeta) = \cdots = G(h\zeta)^n v^0(\zeta)$$ will appear.

Amplification matrix is

$$G^n(h\zeta) = \left(\left(\frac{1}{2} + A\frac{\lambda}{2}\right)e^{ih\zeta} + \left(\frac{1}{2} - A\frac{\lambda}{2}\right)e^{-ih\zeta}\right)^n$$

$$G^n(h\zeta) = \left(I\cos(h\zeta) - iA\lambda\sin(h\zeta)\right)^n .$$

Let us suppose $A = T\Lambda T^{-1}$. The amplification matrix is

$$G^n(h\zeta) = T\left(I\cos(h\zeta) - i\Lambda\lambda\sin(h\zeta)\right)^n T^{-1} .$$

This is why $\left|G^n(h\zeta)\right|$ and $I\cos(h\zeta) - i\Lambda\lambda\sin(h\zeta)$ are bounded. Since $I\cos(h\zeta) - i\Lambda\lambda\sin(h\zeta)$ is a diagonal matrix, every element of it is bounded.

Since the elements of $G(h\zeta)$ are composed of complex numbers, the scheme stability is as follows

$$\left|G(h\zeta)G^*(h\zeta)\right| \le 1$$

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$$\left(I\cos(h\zeta)-i\Lambda\lambda\sin(h\zeta)\right)\left(I\cos(h\zeta)+i\Lambda\lambda\sin(h\zeta)\right)\le 1$$
$$\left(I\cos(h\zeta)\right)^2-\left(i\Lambda\lambda\sin(h\zeta)\right)^2\le 1.$$

Λ is a diagonal matrix and its elements are the same as the eigenvalues of A matrix. In this case

$$F(\zeta)=\cos(h\zeta)-i\omega_i(A)\lambda\sin(h\zeta) \qquad (8)$$
$$\left(\cos(h\zeta)\right)^2-\left(i\omega_i(A)\lambda\sin(h\zeta)\right)^2\le 1$$

is inequality. Here $\omega_i(A)$ values are the eigenvalues of the matrix A.

As a result of the form substitution,

$$\lambda\max_{1\le i\le N}\left|\omega_i(A)\right|\le 1 \qquad (9)$$

a condition of stability.

In this way, we can define the conditions of stability as well as the remaining circuits.

Example 1. The following parameters can be entered in the MathCad system, in which case the stabilization requirement is:

$$\lambda\max_{1\le i\le N}\left|\omega_i(A)\right|\le 1.$$

$$M := 300 \qquad N := 50 \qquad i := \sqrt{-1} \qquad \tau := \frac{1}{M} \qquad h := \frac{1}{N}$$

$$\lambda := \frac{\tau}{h} \rightarrow \frac{1}{6} \qquad A := \begin{pmatrix} -1 & -4 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad E := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

P are eigenvalius of matrix A.

$$P := \text{eigenvals}(A)$$

$$P = \begin{pmatrix} -4.662 \\ -0.468 \\ 4.129 \end{pmatrix} \qquad \begin{aligned} p1 &:= -4.662 \\ p2 &:= -0.468 \\ p3 &:= 4.129 \end{aligned}$$

(9) that the condition of stability is fulfilled in the equation

$$\lambda \cdot |p1| = 0.777 \qquad \lambda \cdot |p2| = 0.078 \qquad \lambda \cdot |p3| = 0.688$$

(8) the graphic representation of the expression

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$$F1(u) := \cos(u) - i \cdot p1 \cdot \lambda \cdot \sin(u)$$
$$F2(u) := \cos(u) - i \cdot p2 \cdot \lambda \cdot \sin(u)$$
$$F3(u) := \cos(u) - i \cdot p3 \cdot \lambda \cdot \sin(u)$$

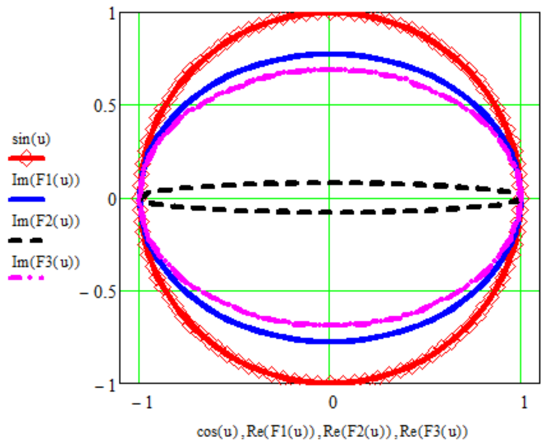

Fig3: Location image in unity circle of $F(\zeta)$ expression.

For the Lax-Friedrichs finite difference scheme $\lambda \max\limits_{1\le i\le N} |\omega_i(A)| \le 1$ condition not fulfilled parameters to the MathCad system.

$$M := 100 \quad N := 50 \quad i := \sqrt{-1} \quad \tau := \frac{1}{M} \quad h := \frac{1}{N}$$

$$P := \mathrm{eigenvals}(A)$$

$$P = \begin{pmatrix} -4.662 \\ -0.468 \\ 4.129 \end{pmatrix} \qquad \begin{aligned} p1 &:= -4.662 \\ p2 &:= -0.468 \\ p3 &:= 4.129 \end{aligned}$$

$$\lambda \cdot |p1| = 2.331 \qquad \lambda \cdot |p2| = 0.234 \qquad \lambda \cdot |p3| = 2.064$$

$\lambda \max\limits_{1\le i\le N} |\omega_i(A)| \le 1$ stability condition can not be fulfilled. If we describe the expression in graphic form (8), then we can say that the graph is out of unity circle. In this case, we see that the Lax-Friedrichs finite difference schemes is unstable.

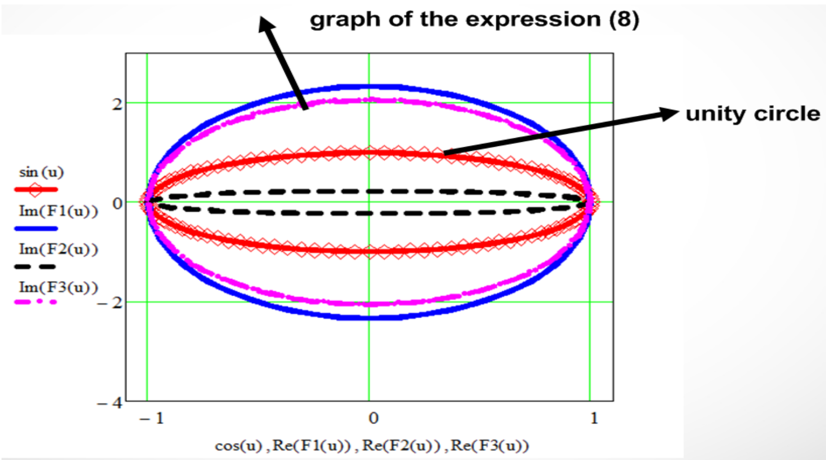

Fig3: Location image out of unity circle of $F(\zeta)$ expression.

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V. CONLLUSIONS

So we can see that it is enough and easier to test $F(\zeta) = \cos(h\zeta) - i\omega_i(A)\lambda\sin(h\zeta)$ the appeared when $\omega_i(A)$ evangelius are put instead of A matrix rather than when testing eigenvalius of $G(\zeta) = I\cos(h\zeta) - iA\lambda\sin(h\zeta)$ application matrix in the unity circle.

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