ISSN: 2350-0328

International Journal of AdvancedResearch in Science, Engineering and Technology

Vol. 5, Issue 11, November 2018

Controllability of a class of composite fractional nonlinear dynamical systems

Haiyong Qin

Lecturer, Doctor, School of Mathematics, Qilu Normal University, Jinan, Shandong, China

ABSTRACT: In this paper, controllability of a class of new composite fractional nonlinear dynamical systems is investigated. Weaker sufficient conditions of controllability for the composite fractional nonlinear dynamical systems are presented. A numerical example is also given to illustrate the main results.

KEY WORDS: Controllability, Fractional composite dynamical systems, Solution, Fixed point theorem.

I. INTRODUCTION

Controllability is one of the most important properties in mathematical control theory, see [1]-[2]. It means that the dynamical systems can be steered from an arbitrary initial state to an arbitrary final state within a limited time by admissible control functions. Venkatesan Govindaraj et al. [1] consider the following composite fractional equation

$$x'(t) + a\,{}^{C}D^{\alpha}x(t) + x(t) = f(t), \quad x(0) = x_0,\ 0 < \alpha < 1,\ t \geq 0. \tag{1.1}$$

The fractional equation (1.1) with order $\alpha = 1/2$ corresponds to a basic problem in fluid dynamics called the Basset problem. So in [1], the authors choose the order value of $\alpha = 1/2$, and the controllability conditions for linear and nonlinear systems are obtained based on the assumptions that the linear systems is controllable. Recently, more and more research is being done on the controllability of fractional dynamical systems by using Grammian matrix, iterative technique and fixed point techniques, see for example, Venkatesan Govindaraj et al.[1], Krishnan Balachandran et al.[2]. The controllability of fractional dynamical systems is one of the most important topics in many problems because the use of fractional derivatives leads to better results that an integer one, see [3]-[6]. The research of the controllability of various types of fractional systems is based on proving the existence of corresponding fractional differential equations, see [3]-[10]. The main difficulty arising in the control problem for nonlinear fractional dynamical systems is the lack of general methods. Venkatesan Govindaraj et al.[1] established sufficient conditions of controllability of the following nonlinear fractional composite dynamical systems

$$\begin{cases} x'(t) + K\,{}^{C}D^{1/2}x(t) + Ax(t) = Bu(t) + f(t, x(t)),\ t \in J \\ x(0) = x_0 \end{cases} \tag{1.2}$$

Inspired by the above literature, in this paper, consider a more generalized form of a class of composite fractional nonlinear dynamical systems

$$\begin{cases} x'(t) + K\,{}^{C}D^{1/2}x(t) + Ax(t) = Bu(t) + f(t, x(t), (Sx)(t)),\ t \in J,\ \ J = [0, T] \\ x(0) = x_0 \end{cases} \tag{1.3}$$

The controllability of nonlinear composite fractional dynamical systems (1.3) is established. The main goal of this paper is to compute a control state that drives the system from a prescribed initial state to a described final state in a limited time. The sufficient conditions of this paper is weaker than the previous work, and an numerical example is provided to illustrate the main results.

II. PRELIMINARIES

This section introduces definitions and preliminaries on fractional calculus. For more details, one can refer to the cited literature and its references.

Definition 2.1 The Caputo fractional derivative of order $\alpha \in \mathbf{C}$ with $n\text{-}1 < \alpha \leq n$, $n \in \mathbf{N}$, for a suitable function f is defined as

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 5, Issue 11, November 2018

$$({}^{C}D^{\alpha}x)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds \tag{2.1}$$

Definition 2.2 Complex parameters α , $\beta \in \mathbf{C}$, the Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \ z\in\mathbf{C},\ \ \alpha,\beta>0 \tag{2.2}$$

$$E_{\alpha,1}(z)=E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \ z\in\mathbf{C},\ \ \alpha,>0 \tag{2.3}$$

For an arbitrary square matrix A, the Mittag-Leffler matrix function is

$$E_{\alpha,\beta}(A)=\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha k+\beta)},\ \ \ \alpha,\beta>0 \tag{2.4}$$

Lemma 2.1 (see,[1])Linear composite fractional dynamical system

$$\begin{cases} x'(t)+K\,{}^{C}D^{1/2}x(t)+Ax(t)=Bu(t),\ t\in J,\ J=[0,T] \\ x(0)=x_0 \end{cases} \tag{2.5}$$

where K, $A\in\mathbf{R}^{n\times n}$, $B\in R^{n\times m}$, $x(t)\in\mathbf{R}^n$, and $u(t)\in L^2(J,\mathbf{R}^m)$. The solution of (2.5) is as defined as following

$$\begin{aligned} x(t)&=(\gamma_1-\gamma_2)^{-1}[\gamma_1E_{1/2}(\gamma_2\sqrt{t})-\gamma_2E_{1/2}(\gamma_1\sqrt{t})]x_0 \\ &\quad+(\gamma_1-\gamma_2)^{-1}\int_0^t(t-s)^{-1/2}[E_{1/2,1/2}(\gamma_1\sqrt{t-s})-E_{1/2,1/2}(\gamma_2\sqrt{t-s})]Bu(s)ds \end{aligned}$$

where $\gamma_1=\dfrac{-K+\sqrt{K^2-4A}}{2}$ and $\gamma_2=\dfrac{-K-\sqrt{K^2-4A}}{2}$, K^2-4A is positive so that the inverse of matrix γ_1 and γ_2 exists.

Definition 2.3 The system (2.5) is controllable on J if, for each vectors $x_0, x_1\in R^n$, there exists a control function $u(t)\in L^2(J,\mathbf{R}^m)$ such that the solution (2.5) with initial state $x(0)=x_0$ satisfies $x(T)=x_1$.

Lemma 2.2(see,[1])The linear composite fractional system (2.5) is controllable on J iffGramianmatix

$$W=\int_0^T N(T-s)BB^*N^*(T-s)ds$$

is positive definite for $T>0$, where $N(t)=(\gamma_1-\gamma_2)^{-1}[E_{1/2,1/2}(\gamma_1\sqrt{t})-E_{1/2,1/2}(\gamma_2\sqrt{t})]$.

III. MAIN RESULTS

Consider nonlinear fractional composite fractional dynamical systems described by the following nonlinear fractional differential equations

$$\begin{cases} x'(t)+K\,{}^{C}D^{1/2}x(t)+Ax(t)=Bu(t)+f(t,x(t),(Sx)(t)),\ t\in J \\ x(0)=x_0 \end{cases} \tag{3.1}$$

where K, $A\in\mathbf{R}^{n\times n}$, $B\in R^{n\times m}$, $x(t)\in\mathbf{R}^n$, and $u(t)\in L^2(J,\mathbf{R}^m)$. $f:J\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous functions. Operator S is defined as following

$$(Sx)(t)=\int_0^t h(t,s)x(s)ds\,, \tag{3.2}$$

where $h(t,s)\in C(J\times J,\mathbf{R}^+)$, and $h_0=\max_{(t,s)\in J\times J}h(t,s)>0$.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 5, Issue 11, November 2018

Let $X=\{x(t)\in C(J,\mathbf{R}^n)$ and ${}^C D^{1/2}x(t)\in C(J,\mathbf{R}^n)\}$ be a Banach space endowed with the norm

$$\|x\|=\max_{t\in J}|x(t)|+\max_{t\in J}|{}^C D^{1/2}x(t)|.$$

In order to obtain the main results, make the following conditions

(H1) There exists a positive M such that

$$\|f(t,x(t),(Sx)(t))\|\le M,\ \forall t\in J,\ x\in\mathbf{R}^n$$

(H2) For $\forall\ \mu_1,\mu_2,\nu_1,\nu_2\in\mathbf{R}^n$, there exist two continuous functions $\varphi(t)$ and $\psi(t)$ on J such that

$$\|f(t,\mu_1,\nu_1)-f(t,\mu_2,\nu_2)\|=\varphi(t)\|\mu_1-\mu_2\|+\psi(t)\|\nu_1-\nu_2\|,\quad t\in J$$

Define $\varphi_0=\max_{t\in J}\varphi(t)$, $\psi_0=\max_{t\in J}\psi(t)$.

For simplicity, Let

$$n_1=\max_{t\in J}\|(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\|$$

$$n_2=\max_{(t,s)\in J\times J}\|(t-s)^{1/2}[\gamma_1E_{1/2}(\gamma_2\sqrt{t-s})-\gamma_2E_{1/2}(\gamma_1\sqrt{t-s})]\|\|B\|$$

$$q=\max_{s\in J}\|(T-s)^{-1/2}N(T-s)\|$$

Theorem 3.1 Suppose that f satisfied conditions (H1) and (H2), and linear systems (2.5) is controllable, then the nonlinear fractional composite fractional dynamical systems (3.1) is controllable on J.

Proof: Defined the following functions

$$x_0(t)=x_0$$

$$x_{n+1}(t)=(\gamma_1-\gamma_2)^{-1}[\gamma_1E_{1/2}(\gamma_2\sqrt{t})-\gamma_2E_{1/2}(\gamma_1\sqrt{t})]x_0+\int_0^t(t-s)^{-1/2}N(t-s)[Bu_n(s)+f(s,x_n(s),(Sx_n)(s))]ds \tag{3.3}$$

$$u_n(t)=(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\left[y_1-\int_0^T(T-s)^{-1/2}N(T-s)f(s,x_n(s),(Sx_n)(s))ds\right] \tag{3.4}$$

where $y_1=x_1-(\gamma_1-\gamma_2)^{-1}[\gamma_1E_{1/2}(\gamma_2\sqrt{t})-\gamma_2E_{1/2}(\gamma_1\sqrt{t})]x_0$.

It is clear that

$$\|u_n(t)\|\le\|(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\|\left[\|y_1\|+\int_0^T\|(T-s)^{-1/2}N(T-s)\|\|f(s,x_n(s),(sx)_n(s))\|ds\right]$$

$$\le\|(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\|\|y_1\|$$

$$+\|(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\|\int_0^T\|(T-s)^{-1/2}N(T-s)\|\|f(s,x_n(s),(Sx_n)(s))\|ds$$

$$\le n_1\|y_1\|+n_1qMT \tag{3.5}$$

and

$$\|u_n(t)-u_{n-1}(t)\|$$

$$\le\|(T-t)^{\frac{1}{2}}B^{*}N^{*}(T-t)W^{-1}\|\left\|\int_0^T\|(T-s)^{-1/2}N(T-s)\|\|f(s,x_n(s),(Sx_n)(s))-f(s,x_{n-1}(s),(Sx_{n-1})(s))\|ds\right\|$$

$$\le n_1q[\int_0^T\|f(s,x_n(s),(Sx_n)(s))-f(s,x_{n-1}(s),(Sx_{n-1})(s))\|ds]$$

$$\le n_1qT\left[\varphi(t)\|x_n(t)-x_{n-1}(t)\|+\psi(t)\|(Sx_n)(t)-(Sx_{n-1})(t)\|\right] \tag{3.6}$$

Since $\|(Sx)_n(t)-(Sx_{n-1})(t)\|\le h_0\int_0^t\|x_n(s)-x_{n-1}(s)\|ds$, it is easy to obtain that

$$\|u_n(t)-u_{n-1}(t)\|\le n_1qT\varphi(t)\|x_n(t)-x_{n-1}(t)\|+n_1qT\psi(t)h_0\int_0^t\|x_n(s)-x_{n-1}(s)\|ds$$

Then

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 5, Issue 11, November 2018

$$\| x_{n+1}(t)-x_n(t) \|$$
$$\leq \int_0^t \|(t-s)^{-1/2} N(t-s)\| [\| B \| \| u_n(s)-u_{n-1}(s) \| + \| f(s,x_n(s),(Sx_n)(s)) - f(s,x_{n-1}(s),(Sx_{n-1})(s)) \|] ds$$
$$= \int_0^t q \| B \| \| u_n(s)-u_{n-1}(s) \| + q \| f(s,x_n(s),(Sx_n)(s)) - f(s,x_{n-1}(s),(Sx_{n-1})(s)) \| ds$$
$$= \int_0^t q \| B \| \| u_n(s)-u_{n-1}(s) \| + q\varphi(t) \| x_n(s)-x_{n-1}(s) \| + q\psi(t) \| (Sx_n)(s)-(Sx_{n-1})(s) \| ds$$
$$\leq \int_0^t q \| B \| \| u_n(s)-u_{n-1}(s) \| + q\varphi_0 \| x_n(s)-x_{n-1}(s) \| + q\psi_0 h_0 \left(\int_0^s \| x_n(\tau)-x_{n-1}(\tau) \| d\tau \right) ds \tag{3.7}$$

Because

$$\| u_1(t)-u_0(t) \| \leq \| (T-t)^{\frac{1}{2}} B^* N^*(T-t) W^{-1} \| \left[\int_0^T \| (T-s)^{-1/2} N(T-s) \| \| f(s,x_1(s),(Sx)_1(s)) - f(s,x_0(s),(Sx_0)(s)) \| ds \right]$$
$$\leq n_1 q [\int_0^T \| f(s,x_1(s),(Sx_1)(s)) - f(s,x_0(s),(Sx_0)(s)) \| ds]$$
$$\leq 2n_1 qMT < L_1 T, L_1 > 0 \tag{3.8}$$

and

$$\| x_1(t)-x_0(t) \| \leq \int_0^t \| (t-s)^{-1/2} N(t-s) \| [\| B \| \| u_0(s) \| + \| f(s,x_0(s),(Sx_0)(s)) \|] ds$$
$$\leq \int_0^t q[\| B \| (n_1 \| y_1 \| + n_1 qMT) + M] ds$$
$$\leq q[\| B \| n_1 \| y_1 \| + n_1 qMT) + M] T < L_2 T, \ L_2 > 0 \tag{3.9}$$

By mathematical induction, we have

$$\| x_{n+1}(t)-x_n(t) \| \leq q \| B \| L_1 \frac{T^{n+1}}{(n+1)!} + q\varphi_0 \frac{L_2^{n+1}}{(n+1)!} + q\psi_0 h_0 \frac{L_2^{n+2}}{(n+2)!} \tag{3.10}$$

Obviously, series $\sum_{n=0}^{\infty} q\|B\| L \frac{T^{n+1}}{(n+1)!}$, $\sum_{n=0}^{\infty} q\varphi_0 \frac{R^{n+1}}{(n+1)!}$ and $\sum_{n=0}^{\infty} q\Psi_0 |h_0| \frac{R^{n+2}}{(n+2)!}$ are convergent. From weierstrass discriminant method, Series $\{x_n(t)\}$ is convergent and uniformly convergent on J. So the following equation can be regarded as the limit of (3.3) and (3.4), respectively.

$$x(t) = (\gamma_1-\gamma_2)^{-1} [\gamma_1 E_{1/2}(\gamma_2 \sqrt{t}) - \gamma_2 E_{1/2}(\gamma_1 \sqrt{t})] x_0 + \int_0^t (t-s)^{-1/2} N(t-s) [Bu(s) + f(s,x(s),(Sx)(s))] ds \tag{3.11}$$

$$u(t) = (T-t)^{\frac{1}{2}} B^* N^*(T-t) W^{-1} \left[y_1 - \int_0^T (T-s)^{-1/2} N(T-s) f(s,x(s),(Sx)(s)) ds \right] \tag{3.12}$$

Meanwhile, we get

$${}^C D^{1/2} x_n(t)$$
$$= \gamma_1\gamma_2(\gamma_1 - \gamma_2)^{-1} [\gamma_1 E_{1/2}(\gamma_2 \sqrt{t}) - \gamma_2 E_{1/2}(\gamma_1 \sqrt{t})] x_0$$
$$+ (\gamma_1 - \gamma_2)^{-1} \int_0^t (t-s)^{-1/2} [\gamma_1 E_{1/2}(\gamma_2 \sqrt{t-s}) - \gamma_2 E_{1/2}(\gamma_1 \sqrt{t-s})] \times [Bu_n(s) + f(s, x_n s(s), (Sx_n)(s))] ds$$

and

$$\| {}^C D^{1/2} x_n(t) - {}^C D^{1/2} x(t) \| \leq \left\| \frac{1}{\gamma_1 - \gamma_2} \int_0^t (t-s)^{-1/2} [\gamma_1 E_{1/2}(\gamma_2 \sqrt{t-s}) - \gamma_2 E_{1/2}(\gamma_1 \sqrt{t-s})] \right.$$
$$\left. \times [B(u_n(s) - u(s)) + (f(s,x_n(s),(Sx_n)(s)) - f(s,x(s),(Sx)(s)))] ds \right\|$$

$$\leq n_2 T \| u_n(t) - u(t) \| + n_2 T \| x_n(t) - x(t) \| + n_2 T \psi_0 h_0 \int_0^t \| x(s) - x_n(s) \| ds \tag{3.13}$$

Then ${}^C D^{1/2} x_n(t) \to {}^C D^{1/2} x(t)$, $n \to \infty$. So (3.11) and (3.12) satisfies equation (3.1) and $x(T) = x_1$, that is, systems (3.1) is controllable on J.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 5, Issue 11, November 2018

IV. NUMERICAL EXAMPLE

In this section, we present an example to illustrate the main results obtained in section III. Consider the following composite nonlinear fractional composite dynamical system

$$\begin{cases} x_1'(t) + \frac{1}{2}\,{}^{C}D^{1/2}x_1(t) - \frac{3}{16}x_1(t) = \sin x_1(t) + \int_0^t x_1(s)ds + u(t) \\ x_2'(t) - {}^{C}D^{1/2}x_1(t) + {}^{C}D^{1/2}x_2(t) - \frac{1}{8}x_1(t) - \frac{1}{4}x_2(t) = x_2^2(t) \end{cases} \tag{4.1}$$

withinitial conditions $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for $t \in [0,1]$.

It is obviously that nonlinear fractional composite system (4.1) is one of special cases of (3.1), where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

$K = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -\frac{3}{16} & 0 \\ \frac{1}{8} & -\frac{1}{4} \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f = \begin{bmatrix} \sin x_1(t) + \int_0^t x_1(s)ds \\ x_2^2(t) \end{bmatrix}$.

After simple calculation, we can see that the Gramian matrix

$$W = \int_0^1 N(1-s)BB^*N^*(1-s)ds = \begin{bmatrix} 26.703763 & -22.675577 \\ -22.675577 & 19.478345 \end{bmatrix}$$

Since $|W| > 0$, linear fractional composite dynamical system (2.5) is controllable on $[0,1]$. The control which steers the initial state $x_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to desired state $x_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ in $[0,1]$ is given by

$$u(t) = (1-t)^{\frac{1}{2}} B^* N^*(1-t)W^{-1}\left[y_1 - \int_0^1 (1-s)^{-1/2} N(1-s) f(s, x(s), (Sx)(s))ds\right]$$

where $y_1 = x_1 - (\gamma_1 - \gamma_2)^{-1}[\gamma_1 E_{1/2}(\gamma_2\sqrt{t}) - \gamma_2 E_{1/2}(\gamma_1\sqrt{t})]x_0$, $\gamma_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$, $\gamma_2 = \begin{bmatrix} -\frac{3}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

V. ACKNOWLEDGEMENTS

This research is supported by a Project of Shandong Province Higher Educational Science and Technology Program(Grant No. J17KB121), Shandong Provincial Natural Science Foundation (Grant No. ZR2016AB04), Foundation for Young Teachers of Qilu Normal University (Grant Nos. 2016L0605, 2015L0603, 2017JX2311 and 2017JX2312), Scientific Research Foundation for University Students of Qilu Normal University (Grant Nos. XS2017L01 and XS2017L05).

REFERENCES

[1] VenkatesanGovindaraj, Krishnan Balachandran, Raju K George, "Numerical Approach for the Controllability of Composite Fractional Dynamical Systems", Journal of Applied Nonlinear Dynamics, 7(1), 2018, pg no: 59-72.
[2] Krishnan Balachandran,VenkatesanGovindaraj, "Numerical Controllability of Fractional Dynamical Systems", Optimization, 2014, http://dx.doi.org/10.1080/02331934.2014.906416.
[3] MihaiBoleantu, "Fractional Dynamical Systems and Applications in Economy", Differential Geometry-Dynamical Systems, 10, 2008, pg no: 62-70.
[4] ZeinabSalamatMostaghim, BehrouzParsaMoghaddam, HosseinSamimiHaghgozar, "Numerical Simulation of Fractional-Order Dynamical Systems in Noisy Environments",Computational and Applied Mathematics, 2018, https://doi.org/10.1007/s40314-018-0698-z.
[5] JinrongWang, Yong Zhou, "Existence and Controllability Results for Fractional Semilinear Differential Inclusions", Nonlinear Analysis: Real World Applications, 12, 2011, pg: 3642-3653.
[6] Krishnan Balachandran, JayakumarKokila, "Constrained Controllability of Fractional Dynamical Systems", Numerical Functional Analysis and Optimization, 34(11), 2013, pg:1187-1205.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 5, Issue 11, November 2018

[7] Jinrong Wang, A.G. Ibrahim, D. O'Regan, Yong Zhou, "Controllability for NoninstantaneousImpulsiveSemilinear Functional Differential Inclusions without Compactness", IndagationesMathematicae, 29, 2018, pg:1362-1392.
[8] Liang Lu, Zhenhai Liu, "Existence and Controllability Results for Stochastic Fractional Evolution Hemivariational Inequalities", Applied Mathematics and Computation, 268, 2015, pg: 1164-1176.
[9] N.I. Mahmudov, "Partial-approximate Controllability of Nonlocal Fractional Evolution Equations via Approximating Method", Applied Mathematics and Computation, 334, 2018, pg: 227-238.
[10] Shengda Liu, Amar Debbouche, JinrongWang,"ILC Method for Solving Approximate Controllability of Fractional Differential Equations with Noninstantaneous Impulses", Journal of Computational and Applied Mathematics, 339, 2018, pg: 343-355.