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Representation of solution for second-order impulsive singular integro-differential equation of mixed type with integral boundary conditions

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ABSTRACT: In this paper, solution for second-order nonlinear impulsive singular equation of mixed type with integral boundary conditions in Banachspaces is investigated. Representation of solution for a boundary value problem of second-order nonlinear impulsive singular integro-differential equation of mixed type is presented. The main result is the basis for further study of existence of this type of equations.

KEY WORDS: Impulsive, Singular, Integral boundary conditions, Representation of solution, Second-order

I. INTRODUCTION

The theory of impulsive differential equations has become an important area of investigation in recent years. It is much richer than the corresponding theory of differential equations, because the structure of its emergence has deep physical background. The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in the phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, see[1-7].On the other hand, heat condition, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Boundary-value problems with integral boundary conditions include two, three, multipoint and nonlocal boundary-value problems as special cases. For more information about the general theory of integral equations and their relation with boundary value problems, one can refer to [4-6]**.**

In the last years, the theory of ordinary differential equations with singularities in abstract spaces has become an important new branch. Zhang and Feng et al. [8] considered the following boundary value problem with integral boundary conditions for second-order nonlinear impulsive integro-differential equation of mixed type in real Banach space

$$\begin{cases} x^{''}(t)+\omega(t)f\big(t,x(t),x^{'}(t),(Tx)(t),(Sx)(t)\big)=\theta,\ t\epsilon J,\ t\neq t_k, \\ \Delta x|_{t=t_k}=I_k\big(x(t_k)\big), \\ \Delta x^{'}|_{t=t_k}=\bar{I}_k\big(x(t_k),x^{'}(t_k)\big),\ \ (k=1,2,3,\cdots,m), \\ x(0)=x(1)=\int_0^1 v(s)\,x(s)ds, \end{cases} \tag{1.1}$$

here$\omega\in C\big(J,[0,+\infty)\big)$, $f\in C(J\times E\times E\times E\times E,E)$, $J=[0,1]$, $0<t_1<t_2<\cdots<t_k<\cdots t_m<1$, $I_k\in C[E,E]$, $\bar{I}_k\in C[E\times E,E]$, $v\in L^1[0,1]$is nonnegative,θ denotes the zero element of Banach space$E$.

Inspired by the above work, in this paper, we shall establish representation of solution for a boundary value problem of second-order nonlinear impulsive singular integro-differential equation of mixed type with integral boundary conditions in Banach space. The main result is the basis for further study of existence of this type of equation. Consider the following boundary value problem with integral boundary conditions for second-order impulsive singularintegro-differential equation of mixed type in a real Banch space

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$$\begin{cases} u''(t) + f\big(t, u(t), (Tu)(t), (Su)(t)\big) = \theta,\ t\epsilon J'_{+}, \\ \Delta u|_{t=t_k} = I_{0k}\big(u(t_k), u'(t_k)\big), \\ \Delta u'|_{t=t_k} = -I_{1k}\big(u(t_k), u'(t_k)\big),\ (k = 1,2,3,\cdots,m), \\ u(0) = u^*,\ u'(1) = \beta \int_0^1 q(s)\, u'(s)ds + v^*, \end{cases} \tag{1.2}$$

where $J = [0,1]$, $J_{+} = (0,1)$, $0 < t_1 < t_2 < \cdots t_m < 1$, $J'_{+} = J_{+} \setminus \{t_1, \cdots, t_m\}$, $\beta \geq 0$, f may besingular at $t = 0, a$ and $x = \theta$ or $x' = \theta$, $I_{ik}(i = 0,1)$ may be singular at $x = \theta$ or $x' = \theta$, θ denotes the zero element of Banach space$E$, u^*, $v^*\epsilon E$, $q\epsilon L^1[0,1]$is nonnegative.$T$and$S$are the linear operators defined as follows

$$(Tu)(t) = \int_0^t k(t,s)u(s)ds,\ (Su)(t) = \int_0^1 h(t,s)u(s)ds, \tag{1.3}$$

where$k \in C[D, R_{+}]$, $h \in C[D_0, R_{+}]$, $D = \{(t,s) \in J \times J: t \geq s\}$, $D_0 = \{(t,s) \in J \times J: 0 \leq t, s \leq 1\}$, $R_{+} = [0, +\infty)$, $R^{+} = (0, +\infty)$, $\Delta u|_{t=t_k}$denotes the jump of $u(t)$ at $t = t_k$,i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where$u(t_k^+)$,$u(t_k^-)$ represent the right and left limits of$t = t_k$ respectively.

II. PRELIMINARIES

In this section, we present some preliminaries that are useful to the proof of our main results.

Let $PC[J,E] = \{u: u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$,and$u(t_k^+)$exists,$k = 1,2,3,\cdots,m\}$. Obviously $PC[J,E]$ is a Banach space with the norm$\| u \|_{PC} = sup_{t\epsilon J} \| u(t) \|$,Let $PC^1[J,E] = \{u: u$ is a map from J into E such that $u(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and$u(t_k^+)u'(t_k^+), u'(t_k^-)$exists$\}$. By virtue of the mean value theorem

$$u(t_k) - u(t_k - h) \in h\,\overline{co}\{u'(t): t_k - h < t < t_k\}(h > 0).$$

It is easy to say that the left derivative $u'_{-}(t_k)$ exists and

$$u'_{-}(t_k) = \lim_{h \to 0^+} h^{-}[u(t_k) - u(t_k - h)] = u'(t_k^-).$$

In the following, $u'(t_k)$ is understood as $u'_{-}(t_k)$, Evidently, $PC^1[J,E]$ is a Banach space with norm $\| u \|_{PC^1} = \max\{\| u \|_{PC}, \| u' \|_{PC}\}$. A map $u \in PC^1[J,E] \cap C^2[J'_{+}, E]$ is called a positive solution of second-order nonlinear impulsive integro-differential equation of mixed type(1.2)if $u^{(i)}(t) > \theta$ for $t \in J$ and $u(t)$ satisfies (1.2).

Throughout this paper, we set

$$(\mathrm{H}_1) 0 \leq \beta \int_0^1 q(s)ds < 1.$$

We shall reduce second-order impulsive singular integro-differential equation of mixed type with integral boundary conditions (1.2) to an impulsive integral equation in$E$. To this end, we firstconsider operator A defined by

$$\begin{aligned}(Au)(t) &= u^* + \alpha_1 v^* t + \alpha_1 \beta t \int_0^1 \int_s^1 q(s) f\big(\tau, u(\tau), u'(\tau), (Tu)(\tau), (Su)(\tau)\big) d\tau ds \\ &+ \left[\int_0^1 t f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)ds - \int_0^t (t-s) f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)ds\right] \\ &+ \alpha_1 \beta t \left[\textstyle\sum_{k=1}^m I_{1k}\big(u(t_k), u'(t_k)\big) \int_0^1 q(s)ds - \int_0^1 q(s) \sum_{0<t_k<s} I_{1k}\big(u(t_k), u'(t_k)\big)\, ds\right] \\ &+ \left[t \sum_{k=1}^m I_{1k}\big(u(t_k), u'(t_k)\big) - \sum_{0<t_k<t} (t - t_k) I_{1k}\big(u(t_k), u'(t_k)\big)\right] + \sum_{0<t_k<t} I_{0k}\big(u(t_k), u'(t_k)\big).\end{aligned} \tag{2.1}$$

III. MAIN RESULTS

Theorem3.1. Assume that(H_1)holds. $u \in PC^1[J,E] \cap C^2[J'_{+}, E]$is a solution of (1) if and only if$u \in PC^1[J,E]$is a solution of the following impulsive integral equation

$$\begin{aligned}u(t) &= u^* + \alpha_1 v^* t + \alpha_1 \beta t \int_0^1 \int_s^1 q(s) f\big(\tau, u(\tau), u'(\tau), (Tu)(\tau), (Su)(\tau)\big) d\tau ds \\ &+ \left[\int_0^1 t f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)ds - \int_0^t (t-s) f\big(s, u(s), u'(s), (Tu)(s), (Su)(s)\big)ds\right]\end{aligned}$$

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$$+\alpha_1\beta t\left[\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big)\int_0^1 q(s)ds-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\big(u(t_k),u'(t_k)\big)\,ds\right]$$
$$+\left[t\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big)-\sum_{0<t_k<t}(t-t_k)I_{1k}\big(u(t_k),u'(t_k)\big)\right]+\sum_{0<t_k<t} I_{0k}\big(u(t_k),u'(t_k)\big).$$

i.e., $u$is a fixed point of operator Ain $PC^1[J,E]$,where$A$ is defined as in (2.1).

Proof. First suppose that$u\in PC^1[J,E]$is a solution of problem (1.2).It is easy to see by integration of (1.2) that

$$u'(t)=u'(0)-\int_0^t f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)ds-\sum_{0<t_k<t} I_{1k}\big(u(t_k),u'(t_k)\big).\quad(2.2)$$

Integrating again, we can get

$$u(t)=u(0)+u'(0)t-\int_0^t(t-s)f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)ds+\sum_{0<t_k<t} I_{0k}\big(u(t_k),u'(t_k)\big)\quad(2.3)$$

Letting$t=1$in (2.2), we find

$$u'(1)=u'(0)-\int_0^1 f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)ds-\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big).\quad(2.4)$$

Substituting$u'(1)=\beta\int_0^1 q(s)u'(s)ds+v^*$into (2.4)

$$u'(0)=\beta\int_0^1 q(s)u'(s)ds+v^*+\int_0^1 f\big(s,u(s),u'(s),(Tu)(s),(Su)(s)\big)ds+\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big),\quad(2.5)$$

where

$$\int_0^1 q(s)u'(s)ds=\int_0^1 q(s)u'(0)ds-\int_0^1\int_0^s q(s)f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)d\tau ds$$
$$-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\big(u(t_k),u'(t_k)\big)ds$$
$$=\int_0^1 q(s)\left[\beta\int_0^1 q(t)u'(\tau)ds+v^*+\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big)\right.$$
$$\left.+\int_0^1 f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)d\tau\right]ds$$
$$-\int_0^1\int_0^s q(s)f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)d\tau ds$$
$$-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\big(u(t_k),u'(t_k)\big)ds$$
$$=\frac{1}{1-\beta\int_0^1 q(s)ds}\left[v^*\int_0^1 q(s)ds+\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big)\int_0^1 q(s)ds\right.$$
$$+\int_0^1\int_s^1 q(s)f\big(\tau,u(\tau),u'(\tau),(Tu)(\tau),(Su)(\tau)\big)d\tau ds$$
$$\left.-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\big(u(t_k),u'(t_k)\big)ds\right].\quad(2.6)$$

Then substitutingu$(0)=u^*$, (2.5) and (2.6) into (2.3), we have

$$u(t)=u^*+\frac{\beta t}{1-\beta\int_0^1 q(s)ds}\left[v^*\int_0^1 q(s)ds+\sum_{k=1}^{m} I_{1k}\big(u(t_k),u'(t_k)\big)\int_0^1 q(s)ds\right.$$

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$$+\int_0^1\int_s^1 q(s)f\left(\tau,u(\tau),u^{'}(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau ds$$

$$\left.-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\left(u(t_k),u^{'}(t_k)\right)ds\right]$$

$$+\mathrm{t}\left[v^*+\int_0^1 f\left(s,u(s),u^{'}(s),(Tu)(s),(Su)(s)\right)ds+\sum_{k=1}^{m} I_{1k}\left(u(t_k),u^{'}(t_k)\right)\right]$$

$$-\int_0^t (t-s)f\left(s,u(s),u^{'}(s),(Tu)(s),(Su)(s)\right)ds+\sum_{0<t_k<t} I_{0k}\left(u(t_k),u^{'}(t_k)\right)$$

$$-\sum_{0<t_k<t} I_{1k}\left(u(t_k),u^{'}(t_k)\right)(t-t_k).$$

and so

$$u(t)=u^*+\alpha_1 v^* t+\alpha_1\beta t\int_0^1\int_s^1 q(s)f\left(\tau,u(\tau),u^{'}(\tau),(Tu)(\tau),(Su)(\tau)\right)d\tau ds$$

$$+\left[\int_0^1 tf\left(s,u(s),u^{'}(s),(Tu)(s),(Su)(s)\right)ds-\int_0^t (t-s)f\left(s,u(s),u^{'}(s),(Tu)(s),(Su)(s)\right)ds\right]$$

$$+\alpha_1\beta t\left[\sum_{k=1}^{m} I_{1k}\left(u(t_k),u^{'}(t_k)\right)\int_0^1 q(s)ds-\int_0^1 q(s)\sum_{0<t_k<s} I_{1k}\left(u(t_k),u^{'}(t_k)\right)ds\right]$$

$$+\left[t\sum_{k=1}^{m} I_{1k}\left(u(t_k),u^{'}(t_k)\right)-\sum_{0<t_k<t}(t-t_k)I_{1k}\left(u(t_k),u^{'}(t_k)\right)\right]+\sum_{0<t_k<t} I_{0k}\left(u(t_k),u^{'}(t_k)\right).$$

Conversely, if $x\in PC^1[J,E]$ is a solution of (2.1), it is easy to proof that $x$satisfied (1.2).

Remark 3.1 Since impulsive differential equations and integral boundary conditions is a hot topic of research in recent years. And in order to obtain the existence results of such type of equations, it is important and interesting to consider the representation of solution for a boundary value problem of second-order nonlinear impulsive singular integro-differential equation of mixed type in Banach space. The main results obtained in this paper have deep physical background.

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