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A comparison between Cramer's Rule and Cramer-Elimination Method to solve systems of linear equations

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ABSTRACT: Cramer's Rule is a 262-year old approach to solving systems of n linear equations in n variables. For a system of 3 linear equations, Cramer's rule requires the computations of the determinants of four 3×3 3 matrices. The method also fails when the determinant of the coefficient matrix is zero.

In this work, we propose the Cramer-Elimination method which consists of solving the first 2 linear equations using Cramer's rule and obtaining expressions of the first and second variables in terms of the third variable. Substituting these expressions in the third equation, we obtain the value of the third variable and the solutions of the other variables are then obtained directly. In this way, we reduce the number of determinants computations to 5.

We show that our proposed method is equivalent to classical Cramer's rule for solving general systems of 3 linear equations and it can be used to solve systems of 3 linear equations when the determinant of the 3×3 coefficient matrix is zero. We show how to apply 2×2 Cramer-Elimination Method for system of *n* linear equations through an example and show its advantage over the classical Cramer rule

KEYWORDS: Cramer's rule, systems of linear equations, Determinants, $\mathbf{2 \times 2}$ Cramer-Elimination Method.

I. INTRODUCTION

A. Cramer's Rule for solving systems of *n* linear equations

Cramer's Rule [2]is usually taught in undergraduate course as a valuable tool to solve systems of linear equations. Cramer's Rule gives an explicit expression for the solution of a system and therefore is theoretically important Klein [6] described a pedagogical approach based upon Cramer's rule. The system of *n* linear equations is given by

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots .. a_{1,n}x_n = b_1 \qquad (1)$$
$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots .. a_{2,n}x_n = b_2 \qquad (2)$$
$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots .. a_{3,n}x_n = b_3 \qquad (3)$$
$$a_{4,1}x_1 + a_{4,2}x_2 + a_{4,3}x_3 + \cdots .. a_{4,n}x_n = b_4 \qquad (4)$$
$$\ldots .. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\ldots .. \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \cdots .. a_{n,n}x_n = b_n \qquad (5)$$

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The system can be written in matrix form $\boldsymbol{AX} = \boldsymbol{B}$ (6) Where

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \cdots\cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \cdots\cdots & a_{2,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} \ldots\ldots & a_{n,n} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$$

Theorem-1 (Cramer's Theorem of linear equations by determinants). *[7]* If the determinant $D = \det A$ of a system of n linear equations given by eq. (6) in the same number of unknowns $x_1 \ldots x_n$ is not zero, the system has precisely one solution. This solution is given by the formula

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots\ldots\ldots x_n = \frac{D_n}{D} \quad (7)$$

Where D_k is the determinant obtained from D by replacing in D the k^{th} column by the column with the elements $b_1, \ldots, b_n$.

Hence if eq. (6) is homogeneous and $D \neq 0$, it has the trivial solution $x_1 = x_2 = \ldots = x_n = 0$. If $D = 0$, the homogeneous system also has nontrivial solutions.

The determinants D and D_k in eq. (7) are given by

$$D = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \cdots\cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \cdots\cdots & a_{2,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} \ldots\ldots & a_{n,n} \end{vmatrix} \quad D_1 = \begin{vmatrix} b_1 & a_{1,2} & a_{1,3} & a_{1,4} \cdots\cdots & a_{1,n} \\ b_2 & a_{2,2} & a_{2,3} & a_{2,4} \cdots\cdots & a_{2,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & a_{n,2} & a_{n,3} & a_{n,4} \ldots\ldots & a_{n,n} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{1,1} & b_1 & a_{1,3} & a_{1,4} \cdots\cdots & a_{1,n} \\ a_{2,1} & b_2 & a_{2,3} & a_{2,4} \cdots\cdots & a_{2,n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n,1} & b_n & a_{n,3} & a_{n,4} \ldots\ldots & a_{n,n} \end{vmatrix}, \ldots\ldots D_n = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \cdots\cdots & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \cdots\cdots & b_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} \ldots\ldots & b_n \end{vmatrix}$$

Some recent developments of using Cramer's rule can be found in [3, 8, 4] and the references therein. Cramer's rule is efficient in solving systems of 2 linear equations.For a general systems of 3 linear equations:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 = b_1 \quad (8)$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 = b_2 \quad (9)$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 = b_3 \quad (10)$$

applying Cramer's rule gives

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$$x_1 = \frac{\begin{vmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

$$= \frac{-a_{2,2}a_{1,3}b_3 + a_{2,2}b_1a_{3,3} + a_{1,2}a_{2,3}b_3 - a_{1,2}b_2a_{3,3} + a_{1,3}a_{3,2}b_2 - b_1a_{3,2}a_{2,3}}{a_{3,2}a_{2,1}a_{1,3} - a_{3,2}a_{1,1}a_{2,3} + a_{3,3}a_{1,1}a_{2,2} - a_{3,3}a_{2,1}a_{1,2} + a_{3,1}a_{1,2}a_{2,3} - a_{3,1}a_{1,3}a_{2,2}}$$

$$x_2 = \frac{\begin{vmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

$$= \frac{a_{3,1}a_{1,2}b_2 - a_{3,1}b_1a_{2,2} + b_3a_{1,1}a_{2,2} - b_3a_{2,1}a_{1,2} - a_{3,2}a_{1,1}b_2 + a_{3,2}a_{2,1}b_1}{a_{3,2}a_{2,1}a_{1,3} - a_{3,2}a_{1,1}a_{2,3} + a_{3,3}a_{1,1}a_{2,2} - a_{3,3}a_{2,1}a_{1,2} + a_{3,1}a_{1,2}a_{2,3} - a_{3,1}a_{1,3}a_{2,2}}$$

$$\boldsymbol{x_3} = \frac{\begin{vmatrix} \boldsymbol{a_{1,1}} & \boldsymbol{a_{1,2}} & \boldsymbol{b_1} \\ \boldsymbol{a_{2,1}} & \boldsymbol{a_{2,2}} & \boldsymbol{b_2} \\ \boldsymbol{a_{3,1}} & \boldsymbol{a_{3,2}} & \boldsymbol{b_3} \end{vmatrix}}{\begin{vmatrix} \boldsymbol{a_{1,1}} & \boldsymbol{a_{1,2}} & \boldsymbol{a_{1,3}} \\ \boldsymbol{a_{2,1}} & \boldsymbol{a_{2,2}} & \boldsymbol{a_{2,3}} \\ \boldsymbol{a_{3,1}} & \boldsymbol{a_{3,2}} & \boldsymbol{a_{3,3}} \end{vmatrix}}$$

$$= \frac{a_{3,1}a_{1,2}b_2 - a_{3,1}b_1a_{2,2} + a_{2,2}a_{1,1}b_3 - a_{1,2}a_{2,2}b_3 - b_2a_{3,2}a_{1,1} + b_1a_{3,2}a_{2,1}}{a_{3,2}a_{2,1}a_{1,3} - a_{3,2}a_{1,1}a_{2,3} + a_{3,3}a_{1,1}a_{2,2} - a_{3,3}a_{2,1}a_{1,2} + a_{3,1}a_{1,2}a_{2,3} - a_{3,1}a_{1,3}a_{2,2}}$$

But Cramer's rule has many disadvantages for system of 3 linear equations as it requires the computations of the determinants of four 3×3 matrices which can be viewed as an enormous task by some students. Also, the method fails when the determinant of the 3×3 coefficient matrix is zero. For systems of 4 or more linear equations, the method is practically useless by hand or using the computer.

We consider this example:

Example 1.

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 8 \\ -x_1 + 2x_2 + x_3 &= 0 \\ x_1 - x_2 - 2x_3 &= 0 \end{aligned} \qquad (11)$$

By Cramer's rule,

$$x_1 = \frac{\begin{vmatrix} 8 & 4 & 6 \\ 0 & 2 & 1 \\ 0 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ -1 & 2 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-24}{-16} = \frac{3}{2}$$

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$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 6 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ -1 & 2 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-8}{-16} = \frac{1}{2}$$

$$x_3 = \frac{\begin{vmatrix} 2 & 4 & 8 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 6 \\ -1 & 2 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-8}{-16} = \frac{1}{2}$$

The system requires the computations of the determinants of four 3×3 matrices which involve 9 second order minors.

For systems of $n > 3$ linear equations, a lot of computations are required using Cramer's rule.

Tirthaji [5] developed the Cross-Multiplication Method in Vedic Mathematics to solve systems of 2 linear equations. The method is an alternate form of Cramer's rule without the explicit calculation of determinants. He also illustrated how to apply his method to system of 3 linear equations. Babajee [1] described the method for solving general system of 3 linear equations and extended it to systems of 4 or more linear equations. Students find it easier to compute the determinant of a 2×2 matrix rather than that of a 3×3 matrix. We now propose solving systems of 3 linear equations by Cramer's rule using the determinants of 2×2 matrices only in the following section.

II. 2 × 2 CRAMER-ELIMINATION METHOD FOR SYSTEMS OF 3 LINEAR EQUATIONS

Generalization

We solve eqs. (8) to (10) using the 2 × 2 Cramer-Elimination Method:

Step 1: We express x_1 and x_2 in terms of x_3 by solving eqs. (8) And (9) using Cramer's rule for systems of 2 equations:

$$x_1 = \frac{\begin{vmatrix} b_1 - a_{1,3}x_3 & a_{1,2} \\ b_2 - a_{2,3}x_3 & a_{2,2} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} = \frac{\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} x_3 - \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}, \qquad (12)$$

$$x_2 = \frac{\begin{vmatrix} a_{1,1} & b_1 - a_{1,3} \\ a_{2,1} & b_2 - a_{2,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} = \frac{-\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} x_3 + \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} \qquad (13)$$

Step 2: Substituting eqs. (12) And (13) in eq. (10) of the system, we have

$$a_{3,1}\left(\frac{\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} x_3 - \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}\right) + a_{3,2}\left(\frac{-\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} x_3 + \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}\right) + a_{3,3}\, x_3 = b_3,$$

$$x_3 = \frac{a_{3,1}\begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix} - a_{3,2}\begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} + b_3\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}{a_{3,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} \qquad (14)$$

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$$x_3 = \frac{\begin{vmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ a_{3,1} & a_{3,2} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

by working the determinant along the 3rd row,

Substituting eq. (14) into (12), we have after

$$x_1 = \frac{b_3 \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2} \frac{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} - \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix} \times \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} - a_{3,3} \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix}}{a_{3,1} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2} \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}$$

(15)

We can verify that

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} - \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix} \times \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \times \begin{vmatrix} b_1 & a_{1,3} \\ b_2 & a_{2,3} \end{vmatrix} \qquad (16)$$

Substituting eq. (16) into eq. (15), we have

$$x_1 = \frac{b_3 \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2} \begin{vmatrix} b_1 & a_{1,3} \\ b_2 & a_{2,3} \end{vmatrix} + a_{3,3} \begin{vmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{vmatrix}}{a_{3,1} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2} \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} = \frac{\begin{vmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

Similarly, substituting eq. (14) into (13), we

$$x_2 = \frac{a_{3,1} \frac{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \times \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} - \begin{vmatrix} a_{1,2} & b_1 \\ a_{2,2} & b_2 \end{vmatrix} \times \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} - b_3 \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3} \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}}{a_{3,1} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2} \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3} \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}}$$

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$$= \frac{a_{3,1}\begin{vmatrix} b_1 & a_{1,3} \\ b_2 & a_{2,3} \end{vmatrix} - b_3\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3}\begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix}}{a_{3,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} - a_{3,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + a_{3,3}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} = \frac{\begin{vmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

Since we are using the determinants of 2×2 matrices to eliminate 2 variables we term them the proposed method as 2×2 Cramer-Elimination Method.

Illustration through examples

We explain two ways of using this method to solve eq. (11) which can be written as:

$$2x_1 + 4x_2 = 8 - 6x_3 \tag{17}$$
$$-x_1 + 2x_2 = -x_3 \tag{18}$$
$$x_1 - x_2 - 2x_3 = 0 \tag{19}$$

Method 1

Step 1: We express x_1 and x_2 in terms of x_3 by solving eqs. (17) and (18) using Cramer's rule for systems of 2 equations:

$$x_1 = \frac{\begin{vmatrix} 8 - 6x_3 & 4 \\ -x_3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 8 & 4 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix} x_3}{4 + 4} = 2 - x_3$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8-6x_3 \\ -1 & -x_3 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 8 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 6 \\ -1 & 1 \end{vmatrix} x_3}{4+4} = = 1 - x_3 \tag{20}$$

Step 2: Substituting eq. (20) in eq. (19) of the system, we have

$$2 - x_3 - (1 - x_3) - 2x_3 = 0$$

Which results in $x_3 = \frac{1}{2}$ Using eq. (20), we have $x_1 = \frac{3}{2}$ and $x_2 = \frac{1}{2}$

We observe that we obtain the value of x_3, we readily obtain the values of x_1 and x_2 since they are already expressed in terms of x_3. In method 1, we use Cramer's rule one times follow by a substitution.

The method 1 is almost similar to the usual substitution method but it avoids much algebraic works as we require only one substitution.

Method 2

Step 1: It is similar to step 1 of method 1.

Step 2: We obtain another expression of x_1 in terms of x_3 by solving eqs. (18) and (19) using Cramer's rule for systems of 2 equations:

$$x_1 = \frac{\begin{vmatrix} -x_3 & 2 \\ 2x_3 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{x_3 - 4x_3}{1-2} = 3x_3 \tag{21}$$

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Equating eqs. (20) and (21), we have $2 - x_3 = 3x_3 \Rightarrow x_3 = 1/2$ and the other solutions are obtained as usual. In method 2, we apply Cramer's rule twice and then we equate.

We observe that method 1 requires fewer computations than method 2 and so we will use 2×2 Cramer-Elimination Method 1. We also observe that our method is quicker to work by hand than classical Cramer's rule because it require the computations of 5 determinants of 2×2 matrices and some algebraic manipulations. We note that our method is advantageous over the Gauss-elimination and LU Decomposition methods because we can avoid back-substitution since the variables are already expressed as a function of other variables.

We illustrate through another example in which we show to which equations it is important to apply the Cramer rule.

Application of 2 × 2 Cramer-Elimination Method when determinant of 3 × 3 coefficient matrix is zero

We illustrate with example:

Example:

$$x_1 + 2x_2 - x_3 = 3 \qquad (22)$$
$$2x_1 + 3x_2 + x_3 = 1 \qquad (23)$$
$$x_1 + 3x_2 - 4x_3 = 8 \qquad (24)$$

Since $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 1 & 3 & -4 \end{vmatrix} = 0$ we cannot use Cramer's rule. We use 2×2 Cramer-Elimination Method to solve this system.

Step 1: We express x_1 and x_2 in terms of x_3 by solving eqs. (22) and (23) using Cramer's rule for systems of 2 equations.

$$x_1 = \frac{\begin{vmatrix} 3+x_3 & 2 \\ 1-x_3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} x_3}{3-4} = -7 - 5x_3 \qquad (25)$$

$$x_2 = \frac{\begin{vmatrix} 1 & 3+x_3 \\ 2 & 1-x_3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} x_3}{-1} = = 5 + 3x_3 \qquad (26)$$

Step 2: Eqs. (25) and (26) satisfy eq. (24) of the system. So letting $x_3 = t, t \in R$, We have $x_1 = -7 - 5t$ and $x_2 = 5 + 3t$ the system infinite soulutions.

III. SYSTEMS OF $N > 3$ LINEAR EQUATIONS

We consider the system of $n > 3$ linear equations in eq. (6). we use the same approach as in [1]. We use 2×2 Cramer-Elimination Method to solve.

$$a_{1,1}x_1 + a_{1,2}x_2 = b_1 - (a_{1,3}x_3 + a_{1,4}x_4 \dots \dots \dots + a_{1,n}x_n),$$
$$a_{2,1}x_1 + a_{2,2}x_2 = b_2 - (a_{2,3}x_3 + a_{2,4}x_4 \dots \dots \dots + a_{2,n}x_n),$$

We obtain

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$$x_1=\frac{\begin{vmatrix}b_1 & a_{2,1}\\ b_2 & a_{2,2}\end{vmatrix}-\begin{vmatrix}a_{1,3} & a_{2,1}\\ a_{2,3} & a_{2,2}\end{vmatrix}x_3-\begin{vmatrix}a_{1,4} & a_{2,1}\\ a_{2,4} & a_{2,2}\end{vmatrix}x_4-\cdots\ldots-\begin{vmatrix}a_{1,n} & a_{2,1}\\ a_{2,n} & a_{2,2}\end{vmatrix}x_n}{\begin{vmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix}},$$

$$x_2=\frac{\begin{vmatrix}a_{1,1} & b_1\\ a_{2,1} & b_2\end{vmatrix}-\begin{vmatrix}a_{1,1} & a_{1,3}\\ a_{2,1} & a_{2,3}\end{vmatrix}x_3-\begin{vmatrix}a_{1,1} & a_{1,4}\\ a_{2,1} & a_{2,4}\end{vmatrix}x_4-\cdots\ldots-\begin{vmatrix}a_{1,1} & a_{1,n}\\ a_{2,1} & a_{2,n}\end{vmatrix}x_n}{\begin{vmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix}}$$

We note in the solution of x_1 the first column is replaced by other columns (second column is fixed) and in the solution of x_2 the second column is replaced by other columns (first column is fixed).
Suppose we obtain eq. (36) as:

$$x_1=f_1(x_3,x_4,\ldots\ldots,x_n)=c_{1,1}+c_{1,2}x_3+c_{1,3}x_4+\cdots\ldots\ldots\ldots+c_{1,n-1}x_n,$$
$$x_2=f_2(x_3,x_4,\ldots\ldots,x_n)=c_{2,1}+c_{2,2}x_3+c_{2,3}x_4+\cdots\ldots\ldots\ldots+c_{2,n-1}x_n,$$

Where $c_{i,j}$ are constants, $i,j=1\ldots n-1$. Substituting eq. (37) into eqs. (3) and (4), we are left with a system of $(n-2)$ linear equations involving the variables $x_3,x_4,\ldots,x_n$:

$a_{3,3}x_3+a_{3,4}x_4=b_3-(a_{3,1}f_1(x_3,x_4,\ldots,x_n)+a_{3,2}f_2(x_3,x_4,\ldots,x_n)+a_{3,5}x_5+a_{3,6}x_6+\cdots.+a_{3,n}x_n$ $a_{4,3}x_3+a_{4,4}x_4=b_4-(a_{4,1}f_1(x_3,x_4,\ldots,x_n)+a_{4,2}f_2(x_3,x_4,\ldots,x_n)+a_{4,5}x_5+a_{4,6}x_6+\cdots.+a_{4,n}x_n$

......

$$a_{n,3}x_3+a_{n,4}x_4=b_n-(a_{n,1}f_1(x_3,x_4,\ldots,x_n)+a_{n,2}f_2(x_3,x_4,\ldots,x_n)+a_{n,5}x_5+a_{n,6}x_6+\cdots.+a_{n,n}x_n \tag{27}$$

We have eliminated 2 variables.

Solving the first two equations of (27), we can obtain

$$x_3=f_3(x_5,x_6,\ldots\ldots,x_n)=c_{3,1}+c_{3,2}x_5+c_{3,3}x_6+\cdots\ldots\ldots\ldots+c_{3,n-1}x_n,$$
$$x_4=f_4(x_5,x_6,\ldots\ldots,x_n)=c_{4,1}+c_{4,2}x_5+c_{4,3}x_6+\cdots\ldots\ldots\ldots+c_{4,n-1}x_n, \tag{28}$$

We are left with a system of $(n-4)$ linear equations involving the variables x_5, x_6, ..., x_n. We have now eliminated 4 variables. So each application of 2×2 Cramer-Elimination Method would eliminate 2 variables. Continuing in this way, if n is odd we can finally obtain a linear equation involving x_n. If n is even we can finally obtain a system of 2 linear equations involving x_{n-1} and x_n which can be solved by 2×2 Cramer-Elimination Method. In short, if n is odd, we must apply the 2×2 Cramer-Elimination Method $\frac{n-1}{2}$ times. If n is even, we must apply the 2×2 Cramer-Elimination Method $\frac{n}{2}$ times. We illustrate our theory through an example of a system of 4 linear equations.

Example :

$$3x_1-4x_2+5x_3-4x_4=12$$
$$x_1-x_2+x_3-2x_4=0$$
$$2x_1+x_2+2x_3+3x_4=52 \tag{29}$$
$$2x_1-3x_2+2x_3-x_4=4 \tag{30}$$

We use 2×2 Cramer-Elimination Method to solve

$$3x_1-4x_2=12-5x_3+4x_4$$
$$x_1-x_2=-x_3+2x_4$$

Table 1: Comparison of Cramer's Rule and 2×2 Cramer-Elimination Method for solving systems of 3 linear equations

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Method	Cramer's Rule	2×2 Cramer-Elimination Method
Example 1	Works	Works
number of 2×2 determinants	9	5
Example 3	Fails	Works
number of 2×2 determinants	-	5
Example 4	Fails	Works
number of 2×2 determinants	-	5

We obtain

$$x_1 = \frac{\begin{vmatrix} 12 & -4 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 5 & -4 \\ 1 & -1 \end{vmatrix} x_3 - \begin{vmatrix} -4 & -4 \\ -2 & -1 \end{vmatrix} x_4}{\begin{vmatrix} 3 & -4 \\ 1 & -1 \end{vmatrix}} = -12 + x_3 + 4x_4$$

$$(31)$$

$$x_2 = \frac{\begin{vmatrix} 3 & 12 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} x_3 - \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} x_4}{\begin{vmatrix} 3 & -4 \\ 1 & -1 \end{vmatrix}} = -12 + 2x_3 + 2x_4$$

Substituting eq. (31) in eqs. (29) and (30) of the system, we have, after simplifications,

$$6x_3 + 13x_4 = 88$$

$$-2x_3 + x_4 = -8 \qquad (32)$$

Applying the 2×2 Cramer-Elimination Method to eq. (32), we have

$$x_3 = \frac{\begin{vmatrix} 88 & 13 \\ -8 & 1 \end{vmatrix}}{\begin{vmatrix} 6 & 13 \\ -2 & 1 \end{vmatrix}} = \frac{88 + 104}{6 + 26} = 6$$

$$x_4 = \frac{\begin{vmatrix} 6 & 88 \\ -2 & -8 \end{vmatrix}}{\begin{vmatrix} 6 & 13 \\ -2 & 1 \end{vmatrix}} = \frac{-48 + 176}{6 + 26} = 4$$

Using eq. (31), we have $x_1 = -12 + 6 + 4(4) = 10$ and $x_2 = -12 + 2(6) + 2(4) = 8$

Table 1 compares Cramer's Rule with 2×2 Cramer-Elimination Method for solving systems of 3 linear equations. It can be observed that the proposed method performs better. The extension for systems for $n > 3$ linear equations is carried out in Appendix A.

IV. CONCLUSION

We have studied the 2×2 Cramer-Elimination Method for solving systems of 3 linear equations. We showed that the method works in cases where classical Cramers rule fails. The method can be easily applied to systems of 3, 4 linear equations and is quicker than classical Cramers rule. The 2×2 Cramer-Elimination Method can be used in

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solving systems of 3 and 4 linear equations in undergraduate courses and compare with classical Cramers rule. For example, students can try the two methods by hand and determine which method is quicker and easier to apply.

REFERENCES

[1] D K R Babajee. Solving systems of linear equations using the Paravartya rule in Vedic Mathematics. *Online International Journal of Vedic Mathematics, http://www.vedicmaths.org/Journal.asp*, 2012.
[2] G Cramer. Introduction l'Analyse des lignes Courbes algbriques. *Geneva: Europeana.*, pages 656–659, 1750.
[3] G M Diaz-Toca, L GonzalezVega, and H Lombardi. Generalizing Cramers Rule: Solving Uniformly Linear Systems of Equations. *SIAM J. Matrix Analysis Applications*, 27:621–637, 2005.
[4] K Habgood and I Arel. A condensation-based application of Cramers rule for solving large-scale linear systems. *Journal of Discrete Algorithms, In Press, doi:10.1016/j.jda.2011.06.007*, 2012.
[5] Krsna Tirthaji Maharaja Jagadguru Swami Sri Bharati. *Vedic Mathematics*. Motilal Banarsidas Publishers Private Limited, Delhi, 13 edition, 2011.
[6] R E Klein. Teaching linear systems theory using Cramer's rule. *IEEE Transactions on Education*, 33:258 – 267, 1990.
[7] E Kreyzig. *Advanced Engineering Mathematics*. Wiley, 9 edition, 2005.
[8] I I Kyrchei. Cramer's rule for quaternionic systems of linear equations. *Journal of Mathematical Sciences*, 155:839–858, 2012.