Numerical Solution of the Inverse Source Reconstruction Problem for the Vortex Equation

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ABSTRACT: This paper considers numerical methods for solving the inverse problem of source identification for the one-dimensional vortex equation. A numerical solution algorithm is developed. When solving the inverse problem, additional conditions were used to determine the source. For numerical modeling of the inverse problem, the finite difference method is used. The obtained numerical results of calculations are presented in graphical form.

KEYWORDS: NAVIER-STOKES EQUATIONS, VORTEX EQUATION, DIRECT PROBLEM, INVERSE PROBLEM, DIFFERENCE SCHEME, WOODS CONDITIONS, SWEEP METHOD.

I.INTRODUCTION

The direct problem of mathematical physics is associated with classical boundary-value problems of mathematical physics and is characterized by the need to find a solution that satisfies a given partial differential equation and with the corresponding additional conditions. In inverse problems, the main equation and additional conditions are not completely specified, but there is some additional information about them. With this formulation of inverse problems of mathematical physics, we are talking about coefficient (the equation is not completely specified - some coefficients of the equation are unknown), boundary (boundary conditions are unknown) and evolutionary (initial condition is not specified) inverse problems of mathematical physics. Inverse problems are often ill-posed in the classical sense. Violation of the requirement of continuous dependence of the solution on the input data is typical. Regularization of an ill-posed problem into the class of well-posed problems is achieved by narrowing the class of feasible solutions.In the theory of inverse problems, mathematical physics and their numerical methods are studied in [1]-[5].Numerical solutions of inverse problems of source recovery in a parabolic equation are considered in detail in [5]-[8].

In this paper, the simplest one-dimensional inverse problem of recovering the sources of the vortex equation is considered [9].

II. INVERSE PROBLEM

Consider the problem of identifying the right-hand side of the one-dimensional vortex equation obtained from the system of Navier-Stokes equations. In one-dimensional, the differential equations for the "vortex-current" are:

$$\frac{\partial \omega}{\partial t} + \psi \frac{\partial \omega}{\partial x} = \nu \frac{\partial^2 \omega}{\partial x^2} + Q(t,x), \ \ 0 < x < 1, \ 0 < t < T, \tag{1}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\omega, \ 0 < x < 1, \ 0 < t \le T. \tag{2}$$

Equations (1), (2) are considered under the following initial and boundary conditions.

$$\psi(0,x) = 0, \ \omega(0,x) = 0, \tag{3}$$

$$\psi(t,0) = \psi(t,1) = 0, \ \frac{\partial \psi(t,0)}{\partial x} = 0, \tag{4}$$

$$\omega(t,0) = -\frac{\partial^2 \psi(t,0)}{\partial x^2}, \ \omega(t,1) = -\frac{\partial^2 \psi(t,1)}{\partial x^2}. \tag{5}$$

The direct problem is formulated in the form (1) - (5).

We will consider the inverse problem, in which the known functions $\omega(x,t)$, $\psi(x,t)$ are required to determine $Q(t,x)$ of equation (1).We will assume that the function $Q(t,x)$ is represented in the form

$$Q(t,x) = Q_0 \, \eta(t)\, \varphi(x), \tag{6}$$

where the function $\varphi(x)$ is assumed to be given, and the unknown is the dependence of the source on time - the function $\eta(t)$ in representation (6). This dependence is restored by additional observation of $\omega(x,t)$ at some internal point $0 < x^* < 1$:

$$\omega(x^*,t) = z(t). \tag{7}$$

We arrive at the simplest identification problem for the right-hand side of the vortex equation (1)-(7).

III. A BOUNDARY VALUE PROBLEM FOR A LOADED EQUATION.

The solution to the identification problem is considered under the following constraints [5, 7]:

1. $\varphi(x^*) \ne 0$,

2. $\varphi(x)$ - sufficient smooth function ($\varphi \in C^2[0,1]$),

3. $\varphi(x) = 0$ at the boundary of the computational domain.

The solution to the inverse problem is sought in the form [5, 7]

$$\omega(t,x) = \theta(t)\varphi(x) + w(t,x), \tag{8}$$

where

$$\theta(t) = \int_0^t \eta(s)\,ds. \tag{9}$$

Substitution of (8), (9) into (1), (2), (6) gives the following equations for $w(x,t)$:

$$\frac{\partial w}{\partial t} + \psi\left[\theta(t)\frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\right] = \nu\left[\theta(t)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right], \ 0 < x < 1, \ 0 < t \le T, \tag{10}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\left[\theta(t)\varphi(x) + w(t,x)\right], \ 0 < x < 1, \ 0 < t \le T. \tag{11}$$

Taking into account representation (8), condition (7) leads to the following representation for the unknown $\theta(t)$:

$$\theta(t) = \frac{1}{\varphi(x^*)}\big[z(t) - w(t, x^*)\big]. \tag{12}$$

Substitution of (12) into (10), (11) gives the desired loaded equations

$$\frac{\partial w}{\partial t} + \psi\frac{\partial w}{\partial x} + \frac{1}{\varphi(x^*)}\big[z(t) - w(t, x^*)\big]\bigg[\psi\frac{\partial \varphi}{\partial x} - \nu\frac{\partial^2 \varphi}{\partial x^2}\bigg] = \nu\frac{\partial^2 w}{\partial x^2}, \tag{13}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{\varphi(x^*)}\big[z(t) - w(t, x^*)\big]\varphi(x) - w(t, x). \tag{14}$$

The boundary condition for $w(t, x)$ has the form:

$$w(t, 0) = -\frac{\partial^2 \psi(t, 0)}{\partial x^2},\ w(t, 1) = -\frac{\partial^2 \psi(t, 1)}{\partial x^2}, \tag{15}$$

$$\psi(t, 0) = 0,\ \psi(t, 1) = 0,\ 0 \le t \le T. \tag{16}$$

From (9) for the auxiliary function $\theta(t)$ we have

$$\theta(0) = 0. \tag{17}$$

This allows us to use the initial condition

$$w(0, x) = \omega(0, x) = 0,\ \ \psi(0, x) = 0,\ \ 0 < x < 1. \tag{18}$$

Thus, inverse problem (1) - (7) is formulated as a boundary value problem for the loaded equation (13) - (18) with representation (9), (12) for an unknown dependence of the source on time.

IV. DIFFERENCE SCHEME.

In the region $\overline{D}$, we introduce a uniform grid, respectively, in the x coordinate and in the time t:

$$\overline{\Omega} = \overline{\Omega_h} \times \Omega_\tau = \Big\{(x_i, t_j),\ x_i = i \cdot h,\ t_j = j \cdot \tau,\ 0 \le i \le N,\ 0 \le j \le M,\ h = 1/N,\ \tau = T/M\Big\}.$$

For simplicity, we will assume that the observation point $x = x^*$ coincides with the inner node $i = k$.

To solve the system of equations (13), (14), we use a purely implicit difference scheme [10]. Approximating (13), (14) we obtain

$$\frac{w_i^{j+1} - w_i^j}{\tau} + \psi_i^j\frac{w_{i+1}^{j+1} - w_{i-1}^j}{2h} + \frac{1}{\varphi_k}\Big[z^{j+1} - w_k^{j+1}\Big]\Big[\psi_i^j(\varphi_x')_i - \nu(\varphi_x'')_i\Big] =$$
$$= \nu\frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{h^2},\ i = 1, 2, ..., N-1, \tag{19}$$

$$\frac{\psi_{i+1}^{j+1} - 2\psi_i^{j+1} + \psi_{i-1}^{j+1}}{h^2} = -\frac{1}{\varphi_k}\Big[z^{j+1} - w_k^{j+1}\Big]\varphi_i - w_i^{j+1},\ i = 1, 2, ..., N-1. \tag{20}$$

In the difference formulation of the problem, we replace boundary conditions (15) with Woods conditions. Approximating (15) - (18), we obtain

$$w_0^{j+1} + \frac{w_1^{j+1}}{2} = \frac{3\big(\psi_0^j - \psi_1^j\big)}{h^2},\ w_N^{j+1} + \frac{w_{N-1}^{j+1}}{2} = \frac{3\big(\psi_N^j - \psi_{N-1}^j\big)}{h^2},\ j = 0, 1, ..., M-1. \tag{21}$$

$$\psi_0^{j+1} = 0,\ \psi_N^{j+1} = 0\ \ j = 0, 1, ..., M-1. \tag{22}$$

$$w_i^0 = \omega_i^0 = 0, \ \psi_i^0 = 0, \ i = 0, 1, 2, ..., N. \tag{23}$$

From the solution of the difference problem (19) - (23) in accordance with (12) we define

$$\theta^{j+1} = \frac{1}{\varphi_k}\left(z^{j+1} - w_k^{j+1}\right), \ j = 0, 1, ..., M-1, \tag{24}$$

supplementing these relations with the condition $0 = 0$. Taking into account (9), for the required dependence of the right-hand side on time, we will use the simplest procedure for numerical differentiation:

$$\eta^{j+1} = \frac{\theta^{j+1} - \theta^{j}}{\tau}, \ j = 0, 1, ..., M-1. \tag{25}$$

V. GRID NON-LOCAL PROBLEM

There are no special problems for solving scheme (19) - (23), despite the fact that the grid problem is non-standard (nonlocal) on a new time layer. Equation (19) at the internal nodes in the form

$$\frac{w_i^{j+1}}{\tau} + \psi_i^j \frac{w_{i+1}^{j+1} - w_{i-1}^{j+1}}{2h} - \nu \frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{h^2} + \frac{1}{\varphi_k}\left(\nu\varphi_{x,i}'' - \psi_i^j \varphi_{x,i}'\right)w_k^{j+1} =$$

$$= \frac{w_i^j}{\tau} + \frac{1}{\varphi_k}z^{j+1}\left[\nu\varphi_{x,i}'' - \psi_i^j \varphi_{x,i}'\right], \tag{26}$$

$$\frac{w_i^{j+1}}{\tau} + \psi_i^j \frac{w_{i+1}^{j+1} - w_{i-1}^{j+1}}{2h} - \nu \frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{h^2} + \frac{1}{\varphi_k}\left(\nu\varphi_{x,i}'' - \psi_i^j \varphi_{x,i}'\right)w_k^{j+1} = g_i^j,$$

$$i = 1, 2, ..., N-1, \ j = 0, 1, ..., M-1,$$

where $g_i^j = \dfrac{w_i^j}{\tau} + \dfrac{1}{\varphi_k}z^{j+1}\left[\nu\varphi_{x,i}'' - \psi_i^j \varphi_{x,i}'\right].$

The solution to system (21), (25) is sought in the form

$$w_i^{j+1} = y_i + w_k^{j+1}u_i, \ i = 0, 1, ..., N. \tag{27}$$

Substitution of (27) into (26) allows us to formulate the following grid problems for auxiliary functions y_i, u_i

$$\frac{y_i}{\tau} + \psi_i^j \frac{y_{i+1} - y_{i-1}}{2h} - \nu \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = g_i^j, \ i = 1, 2, ..., N-1, \tag{28}$$

$$y_0 + \frac{y_1}{2} = \frac{3\left(\psi_0^j - \psi_1^j\right)}{h^2}, \ y_N + \frac{y_{N-1}}{2} = \frac{3\left(\psi_N^j - \psi_{N-1}^j\right)}{h^2}, \tag{29}$$

$$\frac{u_i}{\tau} + \psi_i^j \frac{u_{i+1} - u_{i-1}}{2h} - \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{1}{\varphi_k}\left(\nu\varphi_{x,i}'' - \psi_i^j \varphi_{x,i}'\right) = 0, \tag{30}$$

$$i = 1, 2, ..., N-1,$$

$$u_0 = 0, \ u_N = 0. \tag{31}$$

After that, taking into account representation (27), we find w_k^{j+1}:

$$w_k^{j+1} = \frac{y_k}{1 - u_k}. \tag{32}$$

Grid problems (28), (29) and (30), (31) are standard, and their numerical solution presents no problems.

Equation (28), (30) is reduced to the form:

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -f_{i,}^{j}, \quad i = 1, 2, ..., N-1, \tag{33}$$

$$A_i u_{i-1} - C_i u_i + B_i u_{i+1} = -\overline{f}_{i,}^{j}, \quad i = 1, 2, ..., N-1, \tag{34}$$

where

$$A_i = \frac{\nu\tau}{h^2} + \frac{\tau}{2h}\psi_i^j, \quad B_i = \frac{\nu\tau}{h^2} - \frac{\tau}{2h}\psi_i^j, \quad C_i = \frac{2\nu\tau}{h^2} + 1,$$

$$f_i^j = w_i^j + \frac{\tau}{\varphi_k} z^{j+1}\left[\nu\varphi''_{x,i} - \psi_i^j\varphi'_{x,i}\right], \quad \overline{f}_i^j = -\frac{\tau}{\varphi_k}\left(\nu\varphi''_{x,i} - \psi_i^j\varphi'_{x,i}\right).$$

The algorithm for solving boundary value problems (29), (33) using the sweep method has the form:

$$y_i = \alpha_{i+1}y_{i+1} + \beta_{i+1}, \quad i = N-1, N-2, ..., 1, 0, \tag{35}$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \quad \beta_{i+1} = \frac{A_i\beta_i + f_i^j}{C_i - A_i\alpha_i}, \quad i = 1, 2, ..., N-1, \tag{36}$$

$$y_0 = \alpha_1 y_1 + \beta_1, \quad y_0 = -0.5 y_1 + 3\left(\psi_0^j - \psi_1^j\right)\big/h^2,$$

$$\alpha_1 = -0.5, \quad \beta_1 = 3\left(\psi_0^j - \psi_1^j\right)\big/h^2, \tag{37}$$

$$y_{N-1} = \alpha_N y_N + \beta_N, \tag{38}$$

$$y_N + \frac{y_{N-1}}{2} = 3\left(\psi_N^j - \psi_{N-1}^j\right)\big/h^2, \tag{39}$$

representing (38) in (39) we obtain for y_N the following expression

$$y_N = \left[3\left(\psi_N^j - \psi_{N-1}^j\right)\big/h^2 - 0.5\beta_N\right]\big/\left(1 + 0.5\alpha_N\right), \tag{40}$$

Now, solving the boundary value problems (31), (34) using the sweep method, we obtain the following formulas

$$u_i = \overline{\alpha_{i+1}}u_{i+1} + \overline{\beta_{i+1}}, \quad i = N-1, N-2, ..., 1, 0, \tag{41}$$

$$\overline{\alpha_{i+1}} = \frac{B_i}{C_i - A_i\overline{\alpha_i}}, \quad \overline{\beta_{i+1}} = \frac{A_i\beta_i + \overline{f_i^j}}{C_i - A_i\overline{\alpha_i}}, \quad i = 1, 2, ..., N-1, \tag{42}$$

$$u_0 = \overline{\alpha_1}u_1 + \overline{\beta_1}, \quad u_0 = 0, \quad \overline{\alpha_1} = 0, \quad \overline{\beta_1} = 0, \tag{43}$$

$$u_N = 0. \tag{44}$$

VI. CALCULATION EXAMPLES AND CONCLUSIONS

Here we present the results of calculations performed by the model inverse problem (1) - (7). Let us carry out a quasi-real experiment considering the direct problem (1) - (5) with some given right-hand side. The right side is set by dependencies

$$\varphi(x) = \sin\pi x, \quad 0 \le x \le 1,$$

$$\eta(t) = t^2 + 1, \quad 0 \le t \le T = 0.4,$$

$$Q_0 = 100.$$

The problem is solved numerically on the grid $M = 400$, $N = 21$.

Below are the results of reconstructing such a right-hand side from observations at the point $x^* = 0.5$. Based on the calculation results, the grid function z^j is set.

In the solution of the inverse problem, the mesh function $\varphi(t)$ was perturbed using random inaccuracies.We put

$$z_\delta^j = z^j + 2\delta\left(\sigma^j - \frac{1}{2}\right), \quad j = 0,1,...,M,$$

where σ^j - random function uniformly distributed over the interval.The δ valuespecifiestheleveloferror.

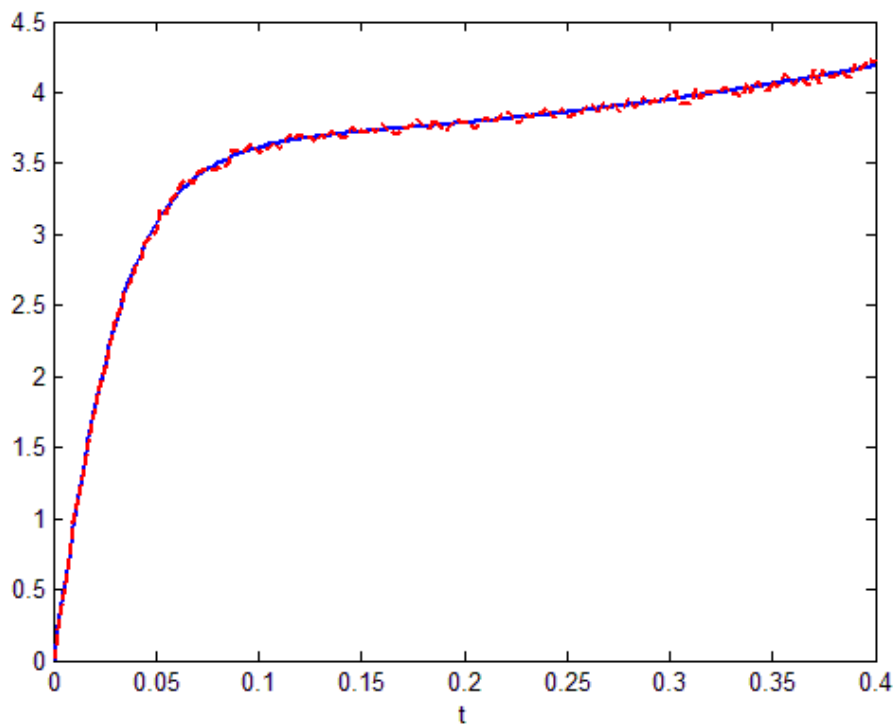Figure 1 shows the functions z_δ and z at an error level of $\delta = 0.0025$.

Figure 1. Accurate and perturbed input data: z and z_δ.

The exact and reconstructed time dependences of the right-hand side of the vortex equation are shown in Figure 2. Figures 3 and 4 show the solutions of the problem obtained with $\delta = 0.01$ and $\delta = 0.001$, respectively.With a decrease in the level of error, the solution is reconstructed more accurately.
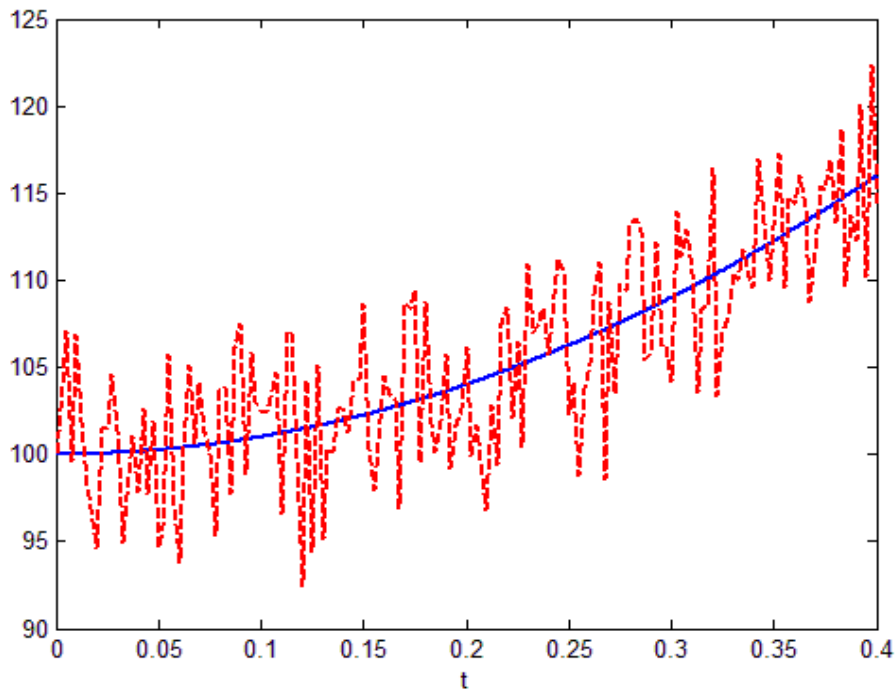
Figure 2. Solution of the inverse problem obtained with $\delta = 0,0025$.
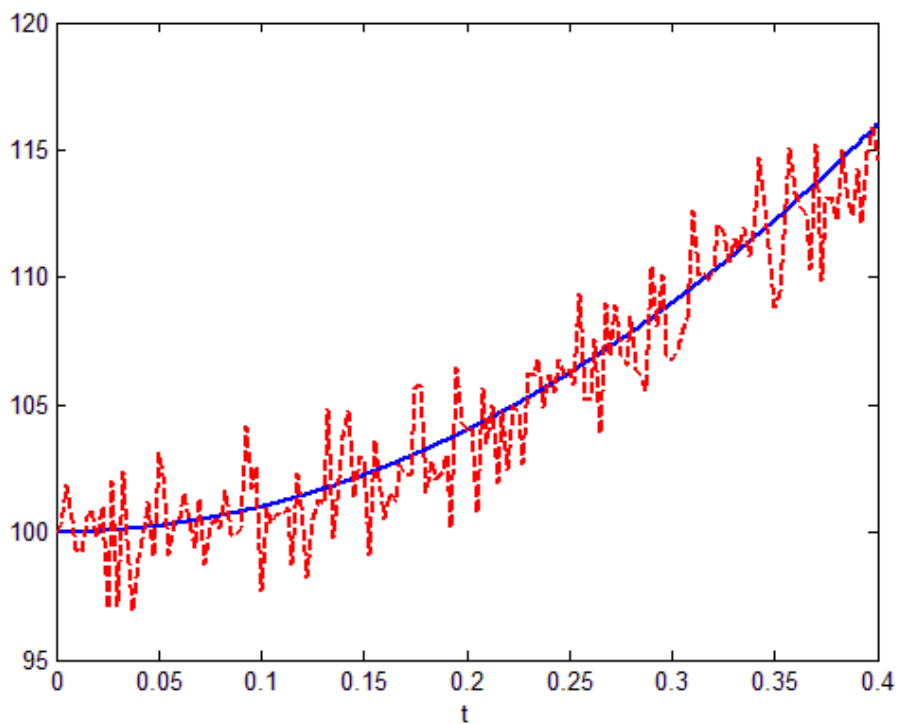
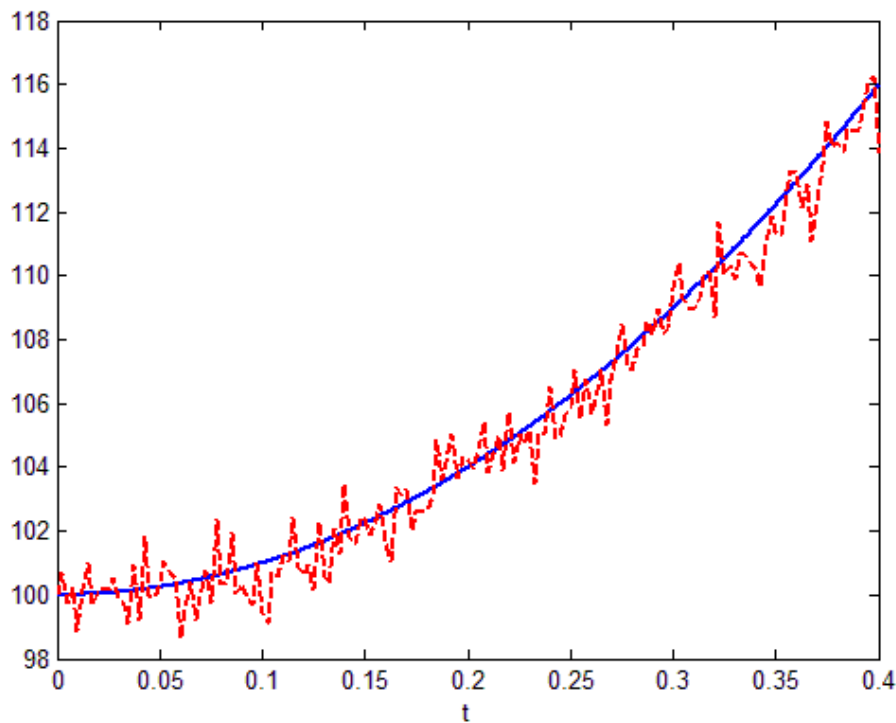Figure 3. Solution of the inverse problem obtained with $\delta = 0,001$.

Figure 4. Solution of the inverse problem obtained with $\delta = 0,0005$.

The considered computational algorithm for solving the inverse problem of the vortex equation can be used to solve more general problems. Particulars, transition to multidimensional tasks, tasks with many observation points, etc.

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