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Application of the Adjoint Symplectic Relation in Viscoelastic Computations

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ABSTRACT: The traditional Saint-Venant problem of three-dimensional viscoelasticity is discussed under the Hamiltonia system with the use of the Laplace integral transformation, and the original problem is transformed into finding eigenvalues and eigenvectors of the Hamiltonia operator matrix. Since local effect near the boundary is usually neglected, all solutions of Saint-Venant problems can be obtained directly by the combinations of zero eigenvectors. Moreover, the adjoint relationships of the symplectic orthogonality of zero eigenvectors in the Laplace domain are generalized to the time domain.

KEY WORDS: Saint-Venant, Eigenvalue, Boundary.

I. INTRODUCTION

A great amount of computational techniques can be found in the research of viscoelastic problems [1]. One of the most popular methods is the use of the Laplace transform, which is an effective method for viscoelasticity, since the equations can be transformed into pseudo-elastic ones. However, this procedure presents some difficulties when viscous parameters vary along time, or when complicated time dependent boundary conditions are imposed. A lot of inverse transforms can not be solved analytically. Therefore the numerical method of the Laplace inverse transformation is rapidly developed and applied. De Chant [2] discussed limitations of the numerical inversion method in the face of discontinuities and asymptotic methods. Temel [3] obtained some solutions in the real space resorting the Durbin's numerical method of the inverse Laplace transform. It should be pointed that numerical inversions of Laplace transform by employing the finite-element and boundary element are effective approximate methods [4]. Because of the complexity of the constitutive relations, it is difficult to find analytical solutions of viscoelasticity, and the numerical method is taken into account in recent years with the help of the rapidly development of the computer technology, especially the finite element method and the boundary element method [5].

The Hamiltonian system is a direct method by which the order of differential governing equations can be reduced. Since the difficulty of solving high-order differential equations in the traditional methods, such as the semi-inverse method, is overcome, the Hamiltonian system gained much attention in recent years and has been applied successfully into elasticity. In this paper, based on the investigation of the character of viscoelastic material, the Hamiltonian system is applied into three-dimensional viscoelasticity, and the dual equations of the system are constructed. Thus the problem is transformed into finding the corresponding eigenvectors, which can easily explain the Sanit-Venant principle: zero eigenvectors are solutions of Sanit-Venant problems, or solutions of the equivalent system, while non-zero eigenvectors are solutions of local effect. By employing the adjoint relationships of the symplectic orthogonality and the expansion of the eigenvectors, effective methods of solving inhomogeneous equations and boundary conditions, especially lateral boundary conditions, are given.

II. SOLUTION METHOD

A homogeneous isotropic viscoelastic cylinder is considered in the Cartesian coordinate $(\tilde{r},\tilde{\theta},\tilde{z})$, in which the z-axis coincides with the centroid axis of the cross section Ω , a single connected domain, and the outward normal $\mathbf{n}$ of its boundary contour $\partial\Omega$ has direction cosines $(l,\ m)$. Let $\tilde{\sigma}_{ij}$ denotes the stress and $\tilde{\varepsilon}_{ij}$ denotes the strain components, then their deviatoric components are as follows

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$$\tilde{S}_{ij} = \tilde{\sigma}_{ij} - \tilde{\sigma}\delta_{ij}, \quad \tilde{S}_{ij} = \tilde{\sigma}_{ij} - \tilde{\sigma}\delta_{ij} \tag{1}$$

where $\tilde{\sigma} = \tilde{\sigma}_{kk}/3$ stand for mean stresses. The constitutive relations of three-dimensional viscoelasticity can be described uniformly as

$$\tilde{S}_{ij} + \eta_m \frac{\partial \tilde{S}_{ij}}{\partial \tilde{t}} = 2G_k \tilde{e}_{ij} + 2\eta G \frac{\partial \tilde{e}_{ij}}{\partial \tilde{t}} \qquad (0 \le G_k \eta_m < \eta G)$$
$$\tilde{\sigma} = 3K\tilde{e} \tag{2}$$

where G is Lame constant, K the bulk modulus, η the viscosity coefficient, η_m and G_k parameters. Eqs. (1) describe the stress-strain relations of viscoelasticity reducible to the standard linear solid type model when $0 < G_k \eta_m < \eta G$, the Kelvin type model when $\eta_m = 0$ and the Maxwell type model when $G_k = 0$. Suppose the lateral boundary is

$$\begin{aligned} l\tilde{\sigma}_{\tilde{r}} + m\tilde{\tau}_{\tilde{r}\tilde{\theta}} &= \tilde{f}^0_{\tilde{r}} \\ l\tilde{\tau}_{\tilde{r}\tilde{\theta}} + m\tilde{\sigma}_{\tilde{\theta}} &= \tilde{f}^0_{\tilde{\theta}} \\ l\tilde{\tau}_{\tilde{r}\tilde{z}} + m\tilde{\tau}_{\tilde{\theta}\tilde{z}} &= \tilde{f}^0_{\tilde{z}} \end{aligned} \tag{3}$$

Let the displacements $u = \tilde{u}/c$, $v = \tilde{v}/c$, $w = \tilde{w}/c$, where $c = p/(2\pi)$ and p is the circumference of the cylinder, and time $t = \tilde{t}/\eta$. Therefore Eqs. (1) are rewritten aswhere

$$S_{ij} + \beta \frac{\partial S_{ij}}{\partial t} = 2\alpha e_{ij} + 2\frac{\partial e_{ij}}{\partial t} \qquad (0 \le \alpha\beta < 1)$$
$$\sigma = 2\gamma e \tag{4}$$

where the dimensionless deviatoric components of stress $S_{ij} = \tilde{S}_{ij}/G$, parameters $\beta = \eta_m/\eta$, $\alpha = G_k/G$, and $\gamma = 3K/(2G)$. Eqs. (3) give the non-dimensional constitutive relations of the standard linear solid type model when $0 < \alpha\beta < 1$, the Kelvin type model when $\beta = 0$ and the Maxwell type model when $\alpha = 0$. Write the displacement variables in vector form:

$$\bar{\mathbf{q}} = \{\bar{u}, \bar{v}, \bar{w}\}^T, \tag{5}$$

Then dual vector can be written as

$$\bar{\mathbf{p}} = \left\{ \begin{array}{c} r\, G^* \left(\dot{\bar{u}} + \dfrac{\partial \bar{w}}{\partial r} \right) \\ G^* \left(\dfrac{\partial \bar{w}}{\partial \theta} + r\dot{\bar{v}} \right) \\ \lambda^* \left(r\dfrac{\partial \bar{u}}{\partial r} + \bar{u} + \dfrac{\partial \bar{v}}{\partial \theta} + \dot{\bar{w}} \right) + 2G^* \dot{\bar{w}} \end{array} \right\} \tag{6}$$

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The Hamiltonian operator matrix can be expressed as

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & \mathbf{-A}^T \end{bmatrix} \quad (7)$$

The components are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial r} \\ 0 & 0 & -\frac{\partial}{r\partial \theta} \\ -a_1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & -a_1\frac{\partial}{r\partial\theta} & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \frac{a_3}{r} & 0 & 0 \\ 0 & \frac{a_3}{r} & 0 \\ 0 & 0 & \frac{a_2}{r} \end{bmatrix} \quad (8)$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial r} \\ 0 & 0 & -\frac{\partial}{r\partial \theta} \\ -a_1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) & -a_1\frac{\partial}{r\partial\theta} & 0 \end{bmatrix} \quad (9)$$

in whchic $a_1 = \lambda^* / (\lambda^* + 2G^*)$, $a_2 = 1/(\lambda^* + 2G^*)$, $a_3 = 1/G^*$, $a_4 = 4G^*(\lambda^* + G^*)/(\lambda^* + 2G^*)$, $a_5 = G^* + 2\lambda^* G^* / (\lambda^* + 2G^*)$ and $a_6 = G^* + a_4$.

III. ADJOINT SYMPLECTIC RELATION

Based on the property of the Hamiltonian operator matrix **H**, the integral product of the eigenvectors is defined as

$$<\overline{\eta}_1, \mathbf{J}, \overline{\eta}_2> = \iint_\Omega \overline{\eta}_1^T \mathbf{J} \overline{\eta}_2 dr d\theta, \quad (10)$$

Thus zero eigenvectors are classified into two groups, which satisfy the adjoint symplectic ortho-normalization relationships:

$$<\overline{\eta}_i^{(\alpha)}, \mathbf{J}, \overline{\eta}_j^{(\beta)}> = -<\overline{\eta}_i^{(\beta)}, \mathbf{J}, \overline{\eta}_j^{(\alpha)}> = \delta_{ij} \quad (11)$$

Introduce another integral product

$$[\eta_1, \mathbf{J}, \eta_2] = \iint_\Omega \eta_1^T \mathbf{J} * \eta_2 dr d\theta = L^{-1} <\overline{\eta}_1, \mathbf{J}, \overline{\eta}_2>, \quad (12)$$

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where * denotes the conventional convolution product. There also exist adjoint symplectic ortho-normalization relationships between the eigenvectors:

$$\begin{aligned} \left[\eta_i^{(\alpha)}, \mathbf{J}, \eta_j^{(\beta)}\right] &= -\left[\eta_i^{(\beta)}, \mathbf{J}, \eta_j^{(\alpha)}\right] = \delta(t)\delta_{ij} \\ \left[\eta_i^{(\alpha)}, \mathbf{J}, \eta_j^{(\alpha)}\right] &= \left[\eta_i^{(\beta)}, \mathbf{J}, \eta_j^{(\beta)}\right] \end{aligned} \tag{13}$$

IV. NUMERICAL EXAMPLE

Consider a circle cylinder and suppose that its both ends are free.The lateral boundary condition is as $\sigma_r = 1 - \cos\theta$ $(r = 1)$. By introducing new variables, the inhomogeneous lateral boundary conditions can be transformed into homogeneous one. However the inhomogeneous term of the dual equations become more complex, and the end condition is changed too. Figs 1-3 exhibit the lateral contours in the bending process of Maxwell type model $(\alpha = 0, \beta = 1)$ at different time: t=0, t=2 and t=4, respectively.

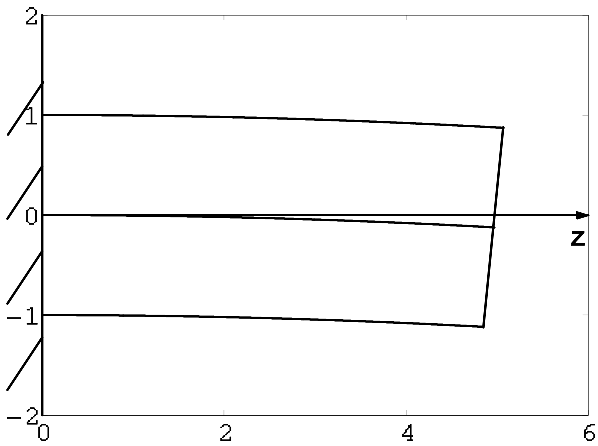

Fig 1. Evolution of the lateral contours of circular cylinder at *t*=0

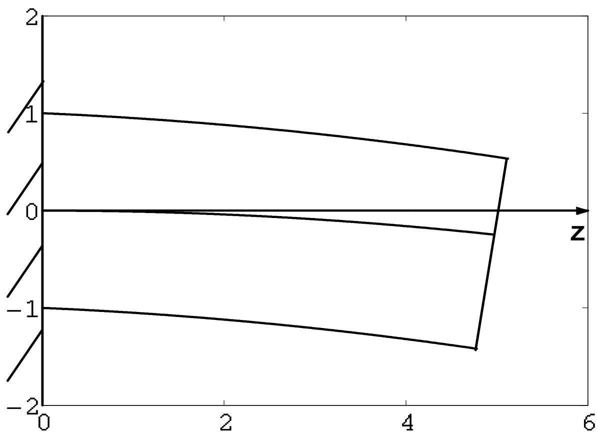

Fig 2. Evolution of the lateral contours of circular cylinder at *t*=2

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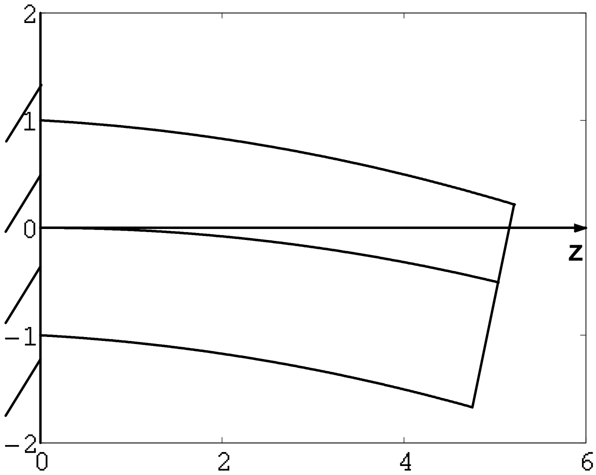

Fig 3. Evolution of the lateral contours of circular cylinder at *t*=4

IV. CONCLUSION AND FUTURE WORK

With the aid of Laplace integral transformation and the property of viscoelasticity, the Hamiltonian system is introduced in the research of three dimensional viscoelasticity. Based on this method, all Saint-Venant solutions and the local effect solutions are obtained from the zero eigenvectors and non-zero eigenvectors. By neglecting the local effect near the boundary, all solutions of Saint-Venant problems can be described approximately by the linear combinations of zero eigenvectors.

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