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Recent Development on General Method of Defining Average: A Brief Outline

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ABSTRACT: In continuation to the three definitions of average developed by Pythagoras namely Arithmetic Mean (*AM*), Geometric Mean (*GM*) & Harmonic Mean (*HM*), a number of definitions of average had already been developed along with some generalized definitions of average. Recently, two general methods have been derived; one for defining the average of a variable and the other for defining the average of a function of a variable. This paper is based on a brief review on the development of these two general methods of defining average.

KEYWORDS: Average, variable, function of a variable, general method of definition

I. INTRODUCTION

Several research had already been done on developing definitions / formulations of average [1 , 2], a basic concept used in developing most of the measures used in analysis of data. Pythagoras [3] is the first mathematician to introduce the concept of average and develop its formulation. He had developed three formulations/definitions of average which were later named as Pythagorean means [4 , 5 , 14 , 18] as a mark of honour to him. The three Pythagorean means are Arithmetic Mean (*AM*), Geometric Mean (*GM*) & Harmonic Mean (*HM*) [4 , 5 , 14 , 18]. A number of definitions/formulations of average have already been developed in continuation to the three Pythagorean means [6 , 7 , 8 , 9 , 10 , 11 , 12 , 13 , 14 , 15 , 16 , 17, 18 , 19]. The next attempt had been initiated towards the development of generalized formulation/definition of average. Kolmogorov [20] formulated one generalized definition of average namely Generalized f – Mean. [7 , 8]. It has been shown that the definitions/formulations of the existing means and also of some new means can be derived from this Generalized f – Mean [9 , 10]. In an study, Chakrabarty formulated one generalized definition of average namely Generalized f_H – Mean [11]. In another study, Chakrabarty formulated another generalized definition of average namely Generalized f_G – Mean [12 , 13] and developed one general method of defining average [15, 16 , 17] as well as the different formulations of average from the first principles [19].

In many real situations, observed numerical data

$$x_1 , x_2 , \ldots\ldots\ldots , x_N$$

are found to be composed of some parameter μ and respective errors

$$\varepsilon_1 , \varepsilon_2 , \ldots\ldots\ldots , \varepsilon_N$$

usually of random in nature i.e

$$x_i = \mu + \varepsilon_i \quad , \quad (i = 1 , 2 , \ldots\ldots\ldots\ldots , N) \qquad \textbf{(1.1)}$$

[21 , 22 , 23 , 24 , 25 , 26 , 27, 28 , 29].

The statistical methods of estimation of the parameter developed so far namely least squares estimation, maximum likelihood estimation, minimum variance unbiased estimation, method of moment estimation and minimum chi-square estimation [31 – 38 , 39 – 40 , 41 – 42 , 43 , 44 , 45 – 47 , 48 – 49 , 50 , 51 – 52], cannot provide appropriate value of the parameter μ [21 , 22 , 23]. Therefore, some methods have recently been developed for determining the value of parameter μ in the situation mentioned above [21 ,22 , 23 , 24 , 25 , 26, 27 , 28 , 29 , 30 , 53 ,54 , 55 , 56, 57 , 58 , 59 , 60]. These methods, however, involve huge computational tasks. Moreover, these methods may not be able to yield the appropriate value of the parameter if observed data used are of relatively small size (and/or of moderately large size too) In reality, of course, the appropriate value of the parameter is not perfectly attainable in practical situation. What one can expect is to obtain that value which is more and more close to the appropriate value of the parameter. Four methods have therefore been developed for determining such value of parameter. These four methods involve lighter load of computational work than respective load involved in the earlier methods and can be applied even if the observed data used are of small size. [61 , 62 , 63 , 64]. The methods developed are based on the concepts of Arithmetic-Geometric Mean (abbreviated as *AGM*) [61 , 62, 67 , 68 , 69 , 70], Arithmetic-Harmonic Mean (abbreviated as *AHM*) [63], Geometric-Harmonic Mean (abbreviated as *GHM*) [64] and Arithmetic-Geometric-Harmonic Mean (abbreviated as

AGHM) [65, 66 , 70] respectively. Each of these four formulations namely *AGM*, *AHM*, *GHM* & *AGHM* has been found to be a measure of parameter μ of the model described by equation (1.1). In other words, each of these four formulations can be regarded as a measure of the central tendency, in addition to the usual measures of central tendency namely *AM*, *GM* & *HM* of the observed values x_1 , x_2 , ……… , x_N , since the values can be expressed by the model (1.1) if μ is the central tendency of them and vice versa. However, for different types of data different measures are suitable.

In this connection, it is to be mentioned that the basis of all the measures of parameter μ of the model described by equation (1.1) or equivalently of the measures of central tendency of x_1 , x_2 , ……… , x_N, is the three Pythagorean means namely Arithmetic Mean (*AM*), Geometric Mean (*GM*) and Harmonic Mean (*HM*) [5 , 14 , 18].

A number of definitions/formulations of average have already been composed by some renowned mathematicians. Some of them, which are often used, are Arithmetic Mean , Geometric Mean , Harmonic Mean , Quadratic Mean , Square Root Mean , Cubic Mean , Cube Root Mean , Generalized p Mean & Generalized p^{th} Root Mean etc. [5 − 19].

In the later stage of development of formulation / definition of average, three generalized definitions / formulations have been constructed by some researchers. These are Generalized *f* - Mean or Generalized f_A – Mean [9 , 10] , Generalized f_H – Mean [11] & Generalized f_G – Mean [12 , 13].

However, none of these three generalized definitions is complete i.e. none of them can describe/yield all types of averages. For this reason, general method has been derived for defining average which is complete i.e. which can describe/yield all types of averages [15 , 16 , 17]. At the first step, one general method has been derived for defining average of the values of a variable [16]. At the next step, one general method has been derived for defining average of a function of a variable [17]. This paper is based on a brief review on the development of these two general methods of defining average.

II. GENERAL METHOD OF DEFINING AVERAGE OF A VARIABLE

Let

$$x_1 , x_2 , \ldots\ldots , x_n$$

be the n values of a variable x .

The arithmetic mean

$$A = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (2.1)$$

satisfies

$$x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n = A + A + \ldots\ldots\ldots\ldots + A$$

This means, the function $f(x_1 , x_2 , x_3 , \ldots\ldots , x_n)$ of $x_1 , x_2 , x_3 , \ldots\ldots , x_n$ defined by

$$f(x_1 , x_2 , x_3 , \ldots\ldots , x_n) = x_1 + x_2 + x_3 + \ldots\ldots + x_n$$

satisfies

$$f(A , A , A , \ldots\ldots\ldots , A) = f(x_1 , x_2 , x_3 , \ldots\ldots , x_n)$$

Here the function $f(x_1 , x_2 , x_3 , \ldots\ldots , x_n)$ is (1) continuous, (2) strictly increasing in each argument of x_1 , $x_2 , x_3 , \ldots\ldots , x_n$ & (3) symmetric (invariant under permutation of the arguments $x_1 , x_2 , x_3 , \ldots\ldots , x_n$.

Similarly, the geometric mean

$$G = \left(\prod_{i=1}^{n} x_i\right)^{1/n} \qquad (2.2)$$

satisfies

$$G.G. \ldots\ldots\ldots\ldots G = x_1 . x_2 . x_3 \ldots\ldots\ldots\ldots x_n$$

This means, the function $g(x_1 , x_2 , x_3 , \ldots\ldots , x_n)$ of $x_1 , x_2 , x_3 , \ldots\ldots , x_n$ defined by

$$g(x_1 , x_2 , x_3 , \ldots\ldots\ldots , x_n) = x_1 . x_2 . x_3 \ldots\ldots\ldots\ldots x_n$$

satisfies

$$g(G , G , G , \ldots\ldots\ldots , G) = g(x_1 , x_2 , x_3 , \ldots\ldots\ldots , x_n)$$

Here, the function $g(x_1 , x_2 , x_3 , \ldots\ldots , x_n)$ is also (1) continuous, (2) strictly increasing in each argument of x_1 , $x_2 , x_3 , \ldots\ldots , x_n$ & (3) symmetric (invariant under permutation of the arguments $x_1 , x_2 , x_3 , \ldots\ldots , x_n$.

Also similarly, the harmonic mean

$$H = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right)^{-1} \qquad (2.3)$$

satisfies

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$$x_1^{-1} + x_2^{-1} + x_3^{-1} + \dots\dots\dots + x_n^{-1} = H^{-1} + H^{-1} + H^{-1} + \dots\dots\dots + H^{-1}$$

This means, the function $h(x_1, x_2, x_3, \dots\dots, x_n)$ of $x_1, x_2, x_3, \dots\dots, x_n$ defined by

$$h(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{-1} + x_2^{-1} + x_3^{-1} + \dots\dots\dots + x_n^{-1}$$

satisfies

$$h(H, H, H, \dots\dots\dots, H) = h(x_1, x_2, x_3, \dots\dots\dots, x_n)$$

In this case also, the function $h(x_1, x_2, x_3, \dots\dots, x_n)$ is (1) continuous, (2) strictly increasing in each argument of $x_1, x_2, x_3, \dots\dots, x_n$ & (3) symmetric (invariant under permutation of the arguments $x_1, x_2, x_3, \dots\dots, x_n$.
Thus, in general, an average of a list

$$x_1, x_2, x_3, \dots\dots, x_n$$

of numbers can be defined to be a number μ such that

$$\phi(\mu, \mu, \mu, \dots\dots\dots, \mu) = \phi(x_1, x_2, x_3, \dots\dots\dots, x_n)$$

where $\phi(x_1, x_2, x_3, \dots\dots\dots, x_n)$ is a function of $x_1, x_2, x_3, \dots\dots, x_n$ such that it is (1) continuous, (2) strictly increasing in each argument of $x_1, x_2, x_3, \dots\dots, x_n$ & (3) symmetric (invariant under permutation of the arguments $x_1, x_2, x_3, \dots\dots, x_n$.
This definition of average can be regarded as a method of deriving various definitions/formulations of average.
Choosing different functions which satisfy the properties (i) , (ii) & (iii), one can obtain different definitions/formulations of average from the equation

$$\phi(\mu, \mu, \mu, \dots\dots\dots, \mu) = \phi(x_1, x_2, x_3, \dots\dots\dots, x_n) \qquad (2.4)$$

II (*a*). Derivation of Various Averages from the Method

Choosing the function $\phi(x_1, x_2, x_3, \dots\dots, x_n)$ as

$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1 + x_2 + x_3 + \dots\dots + x_n,$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1 . x_2 . x_3 \dots\dots\dots . x_n,$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{-1} + x_2^{-1} + x_3^{-1} + \dots\dots\dots + x_n^{-1},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{2} + x_2^{2} + x_3^{2} + \dots\dots\dots + x_n^{2},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{1/2} + x_2^{1/2} + x_3^{1/2} + \dots\dots\dots + x_n^{1/2},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{3} + x_2^{3} + x_3^{3} + \dots\dots\dots + x_n^{3},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{1/3} + x_2^{1/3} + x_3^{1/3} + \dots\dots\dots + x_n^{1/3},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{p} + x_2^{p} + x_3^{p} + \dots\dots\dots + x_n^{p},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = x_1^{1/p} + x_2^{1/p} + x_3^{1/p} + \dots\dots\dots + x_n^{1/p},$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = f(x_1) + f(x_2) + f(x_3) + \dots\dots + f(x_n),$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = f(x_1^{-1}) + f(x_2^{-1}) + f(x_3^{-1}) + \cdots\dots\dots + f(x_n^{-1}),$$
$$\phi(x_1, x_2, x_3, \dots\dots\dots, x_n) = f(x_1) . f(x_2) . f(x_3) \dots\dots\dots f(x_n),$$

respectively and substituting in equation (2.4), the following respective formulations can be obtained :

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i ,$$
$$\mu = \left(\prod_{i=1}^{n} x_i\right)^{1/n} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right)^{-1} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{2}\right)^{1/2} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}\right)^{2} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{3}\right)^{1/3} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1/3}\right)^{3} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{p}\right)^{1/p} ,$$
$$\mu = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1/p}\right)^{p} ,$$
$$\mu = f^{-1}\left\{\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right\} ,$$
$$\mu = f^{-1}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{-1})\right)^{-1}\right\} ,$$
$$\mu = f^{-1}\left[\left\{\prod_{i=1}^{n} f(x_i)\right\}^{1/n}\right] .$$

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*These are respectively arithmetic mean , geometric mean , harmonic mean , quadratic mean , square root mean , cubic mean , cube root mean , Generalized p*th *mean , Generalized p*th root *mean , Generalized f – mean (also called Generalized f_A – mean) , Generalized f_H - mean & Generalized f_G - mean of* $x_1\ ,\ x_2\ , x_3\ ,\ \ldots\ldots ,x_n$.

Note (2.1):

Choosing the function $\phi(x_1\ ,\ x_2\ ,\ x_3\ ,\ \ldots\ldots ,x_n)$ as

$$\phi(x_1\ ,\ x_2\ ,\ x_3\ ,\ldots\ldots\ldots ,x_n) = log\ f(x_1) + log\ f(x_2) + log\ f(x_3) + \ldots\ldots\ldots + log\ f(x_n)$$

one can obtain from equation (2.5) that

$$\mu = f^{-1}\left[antilog\ \{\tfrac{1}{n}\textstyle\sum_{i=1}^{n} log\ f(x_i)\}\right] = f^{-1}[\{(\textstyle\prod_{i=1}^{n} f(x_i)\}^{1/n}$$

This is the definition of Generalized f_G - mean of $x_1\ ,\ x_2\ , x_3\ ,\ \ldots\ldots ,x_n$.

III. GENERAL METHOD OF DEFINING AVERAGE OF A FUNCTION

Now let $y = \xi(.)$ be a function of function so that

$$y_1 = \xi(x_1)\ ,\ y_2 = \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ ,\ y_n = \xi(x_n)$$

are the $\xi(.)$ functional values of

$$x_1\ ,\ x_2\ ,\ \ldots\ldots ,x_n$$

respectively.

Then as per the definition of average, as explained above,

the average of the list

$$y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ ,\ y_n$$

of numbers can be defined to be a number μ such that

$$\phi(\mu\ ,\ \mu\ ,\ \ldots\ldots\ldots ,\mu) = \phi(y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ ,\ y_n) \qquad (3.1)$$

where $\phi(y_1\ ,\ y_2\ ,\ \ldots\ldots\ ,\ y_n)$ is a function of $y_1\ ,\ y_2\ ,\ \ldots\ldots\ ,\ y_n$ which (1) continuous, (2) strictly increasing in each argument of $y_1\ ,\ y_2\ ,\ \ldots\ldots\ ,\ y_n$ & (3) symmetric (invariant under permutation of the arguments $y_1\ ,\ y_2\ ,\ \ldots\ldots\ ,\ y_n$.

This implies that the average of the list

$$\xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)$$

of numbers can be defined to be a number μ such that

$$\phi(\mu\ ,\ \mu\ ,\ \ldots\ldots\ldots ,\mu) = \phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)\} \qquad (3.2)$$

where $\phi\{\ \xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)$ is a function of $\xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)$ which is (1) continuous (2) strictly increasing in each argument of $\xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)$ & (3) symmetric (invariant under permutation of the arguments $\xi(x_1)\ ,\xi(x_2)\ ,\ \ldots\ldots\ldots ,\xi(x_n)$.

III (*a*). Derivation of Various Averages from the Method

Choosing the function $\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\}$ as

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \xi(x_1) + \xi(x_2) + \ldots\ldots\ldots + \xi(x_n)\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \xi(x_1).\xi(x_2).\xi(x_3)\ \ldots\ldots\ldots\ \xi(x_n)\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{-1} + \{\xi(x_2)\}^{-1} + \ldots\ldots\ldots + \{\xi(x_n)\}^{-1}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{2} + \{\xi(x_2)\}^{2} + \ldots\ldots\ldots + \{\xi(x_n)\}^{2}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{1/2} + \{\xi(x_2)\}^{1/2} + \ldots\ldots\ldots + \{\xi(x_n)\}^{1/2}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{3} + \{\xi(x_2)\}^{3} + \ldots\ldots\ldots + \{\xi(x_n)\}^{3}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{1/3} + \{\xi(x_2)\}^{1/3} + \ldots\ldots\ldots + \{\xi(x_n)\}^{1/3}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{p} + \{\xi(x_2)\}^{p} + \ldots\ldots\ldots + \{\xi(x_n)\}^{p},$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = \{\xi(x_1)\}^{1/p} + \{\xi(x_2)\}^{1/p} + \ldots\ldots\ldots + \{\xi(x_n)\}^{1/p}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = f\{\xi(x_1)\} + f\{\xi(x_2)\} + \ldots\ldots\ldots + f\{\xi(x_n)\}\ ,$$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = f_1^{-1} + f_2^{-1} + \ldots\ldots\ldots + f_n^{-1}$$

where $f_1 = f\{\xi(x_1)\}\ ,\ f_2 = f\{\xi(x_2)\}\ ,\ ,\ldots\ldots\ldots ,\ f_n = f\{\xi(x_n)\}\ ,$

$$\phi\{\xi(x_1)\ ,\xi(x_2)\ ,\ldots\ldots\ldots ,\xi(x_n)\} = f\{\xi(x_1)\}.f\{\xi(x_2)\}.\ \ldots\ldots\ldots .f\{\xi(x_n)\}\ ,$$

respectively and substituting in equation (3.2), the following respective formulations can be obtained :

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \xi(x_i)\ ,$$

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$$\mu = \{\prod_{i=1}^{n} \xi(x_i)\}^{1/n} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{-1}]^{-1} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{2}]^{1/2} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{1/2}]^{2} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{3}]^{1/3} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{1/3}]^{3} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{p}]^{1/p} ,$$
$$\mu = [\frac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{1/p}]^{p} ,$$
$$\mu = f^{-1}[\{\frac{1}{n}\sum_{i=1}^{n} f\{\xi(x_i)\}] ,$$
$$\mu = f^{-1}\{(\frac{1}{n}\sum_{i=1}^{n} f_i^{-1})^{-1}\} ,$$
$$\mu = f^{-1}[\{\frac{1}{n}\sum_{i=1}^{n} f\{\xi(x_i)\}] ,$$

where $f_1 = f\{\xi(x_1)\}$, $f_2 = f\{\xi(x_2)\}$, , ……….. , $f_n = f\{\xi(x_n)\}$.
These are respectively arithmetic mean , geometric mean , harmonic mean , quadratic mean , square root mean , cubic mean , cube root mean , Generalized p^{th} *mean , Generalized* p^{th} root *mean , Generalized f – mean (also called Generalized* f_A *– mean) , Generalized* f_H *- mean & Generalized* f_G *- mean of* $\xi(x_1)$, $\xi(x_2)$, ……….. , $\xi(x_n)$.

Note (3.1):

Choosing the function $\phi\{\xi(x_1), \xi(x_2), \ldots, \xi(x_n)\}$ as

$$\phi(x_1 , x_2 , \ldots , x_n) = log f\{\xi(x_1)\} + \log f\{\xi(x_2)\} + \ldots + log f\{\xi(x_n)\}$$

one can obtain from equation (3.7) that

$$\mu = f^{-1}[antilog(\frac{1}{n}\sum_{i=1}^{n} log f_i)\} = f^{-1}[\{\frac{1}{n}\sum_{i=1}^{n} f\{\xi(x_i)\}]$$

which is nothing but the generalized f_G *- mean of* $\xi(x_1)$, $\xi(x_2)$, ……….. , $\xi(x_n)$.

IV. CONCLUSION

The two general methods of defining average, as described above, are based on the principle behind the concept of average. These two general definitions capture the important property of all averages that the average of a list of identical elements is that element itself.

The existing definitions/formulations as well as various new definitions/formulations of average can be derived from theses two general methods defining selecting different forms of the function $\phi(x_1 , x_2 , \ldots , x_n)$.
Thus the equation

$$\phi(\mu , \mu , \mu , \ldots , \mu) = \phi(x_1 , x_2 , x_3 , \ldots , x_n)$$

can be regarded as the generating equation of average of a variable.

Similarly, the equation

$$\phi(\mu , \mu , \ldots , \mu) = \phi\{\xi(x_1), \xi(x_2), \ldots, \xi(x_n)\}$$

can be regarded as the generating equation of average of a function of a variable.

As per the meaning of research [71 , 72 ,73], it can be concluded that the innovations of these two general methods of defining average can be regarded as original/fundamental research carrying very high significance in the theory of average.

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REFERENCES

1. Bakker Arthur (2003): "The early history of average values and implications for education", Journal of Statistics Education, 11(1), 17 – 26.
2. Miguel de Carvalho (2016): "Mean, what do you Mean?", The American Statistician , 70, 764 – 776.
3. Christoph Riedweg (2005): "Pythagoras: his life, teaching, and influence (translated by Steven Rendall in collaboration with Christoph Riedweg and Andreas Schatzmann, Ithaca)", ISBN 0-8014-4240-0, Cornell University Press.
4. David W. Cantrell: "Pythagorean Means", Math World.
5. Dhritikesh Chakrabarty (2016): "Pythagorean Mean: Concept behind the Averages and Lot of Measures of Characteristics of Data", NaSAEAST- 2016, Abstract ID: CMAST_NaSAEAST (Inv)-1601). https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
6. Dhritikesh Chakrabarty (2017): "Objectives and Philosophy behind the Construction of Different Types of Measures of Average", NaSAEAST-2017, Abstract ID: CMAST_NaSAEAST (Inv)- 1701). https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
7. Kolmogorov Andrey (1039): "On the Notion of Mean", in "Mathematics and Mechanics" (Kluwer 1991), 144 – 146.
8. Kolmogorov Andrey (1033): "Grundbegriffe der Wahrscheinlichkeitsrechnung (in German), Berlin: Julius Springer.
9. Dhritikesh Chakrabarty (2018): "Derivation of Some Formulations of Average from One Technique of Construction of Mean", American Journal of Mathematical and Computational Sciences, 2018, 3(3), 62 – 68. http://www.aascit.org/journal/ajmcs.
10. Dhritikesh Chakrabarty (2018): "One Generalized Definition of Average: Derivation of Formulations of Various Means", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 7(3), 212 – 225. www.jecet.org .
11. Dhritikesh Chakrabarty (2018): "f_H -Mean: One Generalized Definition of Average", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 7(4), 301 – 314. www.jecet.org.
12. Dhritikesh Chakrabarty (2018): "Generalized f_G - Mean: Derivation of Various Formulations of Average", American Journal of Computation, Communication and Control, 2018, 5(3), 101 – 108. http://www.aascit.org/journal/ajmcs .
13. Dhritikesh Chakrabarty (2019): "One Definition of Generalized f_G - Mean: Derivation of Various Formulations of Average", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C , 2019, 8(2), 051 – 066. www.jecet.org .
14. Dhritikesh Chakrabarty (2016): "Pythagorean Mean: Concept behind the Averages and Lot of Measures of Characteristics of Data", NaSAEAST-2016, Abstract ID: CMAST_NaSAEAST (Inv)-1601), 2016. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
15. Dhritikesh Chakrabarty (2018): "General Technique of Defining Average", NaSAEAST- 2018, Abstract ID: CMAST_NaSAEAST -1801 (I). https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
16. Dhritikesh Chakrabarty (2019): "One General Method of Defining Average: Derivation of Definitions/Formulations of Various Means", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 8(4), 327 – 338. www.jecet.org .
17. Dhritikesh Chakrabarty (2019): "A General Method of Defining Average of Function of a Set of Values", Aryabhatta Journal of Mathematics & Informatics, 11(2), 269 – 284. www.abjni.com .
18. Dhritikesh Chakrabarty (2019): "Pythagorean Geometric Mean: Measure of Relative Change in a Group of Variables", NaSAEAST- 2019, Abstract ID: CMAST_NaSAEAST -1902 (I), 2019. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
19. Dhritikesh Chakrabarty (2020): "Definition / Formulation of Average from First Principle", Journal of Environmental Science, Computer Science and Engineering & Technology, Section C, 9(2), 151 – 163. www.jecet.org .
20. Youschkevitch A. P (1983): "A. N. Kolmogorov: Historian and philosopher of mathematics on the occasion of his 80th birfhday", Historia Mathematica , 10(4), 383 – 395.
21. Dhritikesh Chakrabarty (2014): "Determination of Parameter from Observations Composed of Itself and Errors", International Journal of Engineering Science and Innovative Technology, 2014, 3(2), 304 – 311. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
22. Dhritikesh Chakrabarty (2014): "Analysis of Errors Associated to Observations of Measurement Type", International Journal of Electronics and Applied Research , 1(1), 15 – 28. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
23. Dhritikesh Chakrabarty (2014): "Observation Composed of a Parameter and Random Error: An Analytical Method of Determining the Parameter", International Journal of Electronics and Applied Research , 1(2), 20 – 38 . https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
24. Dhritikesh Chakrabarty (2015): "Observation Consisting of Parameter and Error: Determination of Parameter", Proceedings of the World Congress on Engineering , ISBN: 978-988-14047-0-1, ISSN: 2078 - 0958 (Print); ISSN: 2078-0966 (Online). https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
25. Dhritikesh Chakrabarty (2015):"Observation Consisting of Parameter and Error: Determination of Parameter," Lecture Notes in Engineering and Computer Science (ISBN: 978-988-14047-0-1), London, 2015, 680 – 684. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.
26. Dhritikesh Chakrabarty (2015): "Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati", J. Chem. Bio. Phy. Sci. , Sec. C , 5(3), 2863 – 2877. Available at: www.jcbsc.org .
27. Dhritikesh Chakrabarty (2015): Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati Based on Midrange and Median", J. Chem. Bio. Phy. Sci. Sec. D, 5(3), 3193 – 3204. www.jcbsc.org .
28. Dhritikesh Chakrabarty (2015): "Observation Composed of a Parameter and Random Error: Determining the Parameter as Stable Mid Range", International Journal of Electronics and Applied Research 2(1), 35 – 47. http://eses.net.in/ESESJournal .
29. Dhritikesh Chakrabarty (2015): "A Method of Finding Appropriate value of Parameter from Observation Containing Itself and Random Error", Indian Journal of Scientific Research and Technology, 3(4), 14 – 21. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
30. Dhritikesh Chakrabarty (2015): "Theoretical Model Modified For Observed Data: Error Estimation Associated To Parameter", International Journal of Electronics and Applied Research , 2(2), 29 – 45. http://eses.net.in/ESESJournal .
31. Hald Anders (1999): "On the History of Maximum Likelihood in Relation to Inverse Probability and Least Squares", Statistical Science , 14, 214 – 222.
32. Barnard G. A. (1949): "Statistical Inference", Journal of the Royal Statistical Society, Series B, 11, 115 – 149.
33. Birnbaum Allan (1962): "On the Foundations of Statistical Inference", Journal of the American Statistical Association , 57, 269 – 306.
34. Ivory (1825): "On the Method of Least Squares", Phil. Mag., LXV, 3 – 10.
35. Kendall M. G. and Stuart A(1977): , "Advanced Theory of Statistics", Vol. 1 & 2, 4th Edition, New York, Hafner Press .
36. Lehmann Erich L. & Casella George (1998): "Theory of Point Estimation", 2nd ed. Springer. ISBN 0 – 387 – 98502 – 6 .

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 8, Issue 8 , August 2021

37. Lucien Le Cam (1990): "Maximum likelihood — An introduction", ISI Review, 8(2), 153 – 171.
38. Walker Helen M. & Lev J. (1965): "Statistical Inference", Oxford & IBH Publishing Company.
39. Dhritikesh Chakrabarty & Atwar Rahman (2007): "Exponential Curve : Estimation Using the Just Preceding Observation in Fitted Curve", Int. J. Agricult. Stat. Sci. , 3(2), 381 – 386. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
40. Dhritikesh Chakrabarty & Atwar Rahman (2008): "Gompartz Curve : Estimation Using the Just Preceding Observation in Fitted Curve", Int. J. Agricult. Stat. Sci. , 4(2), 421 – 424. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.
41. Atwar Rahman & Dhritikesh Chakrabarty (2009): "Linear Curve : A Simpler Method of Obtaining Least squares Estimates of Parameters", Int. J. Agricult. Stat. Sci. , 5(2), 415 – 424. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
42. Dhritikesh Chakrabarty (2011): "Finite Difference Calculus: Method of Determining Least Squares Estimates", AryaBhatta J. Math. & Info . 3(2), 363 – 373. www.abjni.com .
43. Atwar Rahman & Dhritikesh Chakrabarty (2011): "General Linear Curve : A Simpler Method of Obtaining Least squares Estimates of Parameters", Int. J. Agricult. Stat. Sci. , 7(2), 429 – 440. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
44. Dhritikesh Chakrabarty (2014): "Curve Fitting: Step-Wise Least Squares Method", AryaBhatta J. Math. & Info. , 6(1), 15 – 24. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
45. Atwar Rahman & Dhritikesh Chakrabarty (2015): "Elimination of Parameters and Principle of Least Squares: Fitting of Linear Curve to Average Minimum Temperature Data in the Context of Assam", International Journal of Engineering Sciences & Research Technology, 4(2), 255 – 259. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
46. Atwar Rahman & Dhritikesh Chakrabarty (2015): "Elimination of Parameters and Principle of Least Squares: Fitting of Linear Curve to Average Maximum Temperature Data in the Context of Assam", AryaBhatta J. Math. & Info. (ISSN (Print): 0975 – 7139, 7(1), 23 – 28. www.abjni.com .
47. Atwar Rahman & Dhritikesh Chakrabarty (2015): "Basian-Markovian Principle in Fitting of Linear Curve", The International Journal Of Engineering And Science, 4(6), 31 – 43. www.theijes.com .
48. Ivory, On the Method of Least Squares, *Phil. Mag.*, LXV, 3 – 10, 1825.
49. Kendall M. G. and Stuart A, Advanced Theory of Statistics, *Vol. 1 & 2, 4th Edition, New York, Hafner Press*, 1977.
50. Atwar Rahman & Dhritikesh Chakrabarty (2015): "Method of Least Squares in Reverse Order: Fitting of Linear Curve to Average Minimum Temperature Data at Guwahati and Tezpur", AryaBhatta J. Math. & Info . 7(2), 305 – 312. Available at www.abjni.com .
51. Dhritikesh Chakrabarty (2016): "Elimination-Minimization Principle: Fitting of Polynomial Curve to Numerical Data", International Journal of Advanced Research in Science, Engineering and Technology, 3(5), 2067 – 2078. www.ijarset.com .
52. Dhritikesh Chakrabarty (2016): "Elimination-Minimization Principle: Fitting of Exponential Curve to Numerical Data", International Journal of Advanced Research in Science, Engineerin and Technology, 3(6), 2256 – 2264. www.ijarset.com .
53. Dhritikesh Chakrabarty (2016): "Impact of Error Contained in Observed Data on Theoretical Model: Study of Some Important Situations", International Journal of Advanced Research in Science, Engineering and Technology, 3(1), 1255 – 1265. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats .
54. Dhritikesh Chakrabarty (2016): "Theoretical Model and Model Satisfied by Observed Data: One Pair of Related Variables", International Journal of Advanced Research in Science, Engineering and Technology, 3(2), 1527 – 1534. www.ijarset.com .
55. Dhritikesh Chakrabarty (2017): "Variable(s) Connected by Theoretical Model and Model for Respective Observed Data", FSDM2017, Abstract ID: FSDM2220, 2017. https://www.researchgate.net/profile/Dhritikesh_Chakrabarty/stats.
56. Dhritikesh Chakrabarty (2017): "Numerical Data Containing One Parameter and Random Error: Evaluation of the Parameter by Convergence of Statistic", International Journal of Electronics and Applied Research , 4(2), 59 – 73. http://eses.net.in/ESESJournal .
57. Dhritikesh Chakrabarty (2018): "Observed Data Containing One Parameter and Random Error: Evaluation of the Parameter Applying Pythagorean Mean", International Journal of Electronics and Applied Research , 5(1), 32 – 45. http://eses.net.in/ESESJournal .
58. Dhritikesh Chakrabarty (2019): "Significance of Change of Rainfall: Confidence Interval of Annual Total Rainfall", Journal of Chemical, Biological and Physical Sciences, Sec. C, 9(3), 151 – 166. www.jcbsc.org .
59. Dhritikesh Chakrabarty (2019): "Observed Data Containing One Parameter and Random Error: Probabilistic Evaluation of Parameter by Pythagorean Mean", International Journal of Electronics and Applied Research, 6(1), 24 – 40. http://eses.net.in/ESESJournal .
60. Dhritikesh Chakrabarty (2019): "Significance of Change in Surface Air Temperature in the Context of India", Journal of Chemical, Biological and Physical Sciences, Sec. C, 9(4), 251 – 261. www.jcbsc.org .
61. Dhritikesh Chakrabarty (2019): "Arithmetic-Geometric Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error", International Journal of Electronics and Applied Research , 6(2), 98 – 111. http://eses.net.in/ESESJournal .
62. Dhritikesh Chakrabarty (2020): "AGM: A Technique of Determining the Value of Parameter from Observed Data Containing Itself and Random Error", Journal of Environmental Science, Computer Science nd Engineering & Technology, Section C, 9(3), 2020, 473 – 486. www.jecet.org .
63. Dhritikesh Chakrabarty (2020): "Arithmetic-Harmonic Mean: Evaluation of Parameter from Observed Data Containing Itself and Random Error", International Journal of Electronics and Applied Research 7(1), 29 – 45 . http://eses.net.in/ESESJournal .
64. Dhritikesh Chakrabarty (2020): "Determination of the Value of Parameter μ of the Model $X = \mu + \varepsilon$ by GHM", International Journal of Advanced Research in Science, Engineering and Technology, 7(11), 15801 – 15810 . www.ijarset.com .
65. Dhritikesh Chakrabarty (2020): "Central Tendency of Annual Extremum of Surface Air Temperature at Guwahati by AGHM", International Journal of Advanced Research in Science, Engineering and Technology, 7(12), 16088 – 16098. www.ijarset.com .
66. Dhritikesh Chakrabarty (2021): "AGM, AHM, GHM & AGHM: Evaluation of Parameter μ of the Model $X = \mu + \varepsilon$ ", International Journal of Advanced Research in Science, Engineering and Technology, 8(2), 16691 – 16699 . www.ijarset.com .
67. *David A. Cox (2004): "The Arithmetic-Geometric Mean of Gauss", In J. L. Berggren; Jonathan M. Borwein; Peter Borwein (eds.).* Pi: A Source Book. *Springer. p. 481.* ISBN 978-0-387-20571-7 , (first published in L'Enseignement Mathématique, t. 30 (1984), p. 275 – 330).
68. Hazewinkel, Michiel *ed. (2001):* "Arithmetic–geometric mean process, Encyclopedia of Mathematics", *Springer Science+Business Media B.V. / Kluwer Academic Publishers,* ISBN 978-1-55608-010-4 .
69. Dhritikesh Chakrabarty (2021): "Comparison of Measures of Parameter of the Model $X = \mu + \varepsilon$ Based On Pythagorean Means", International Journal of Advanced Research in Science, Engineering and Technology, 8(3), 16948 – 16956, www.ijarset.com .
70. Dhritikesh Chakrabarty (2021): "Four Formulations of Average Derived from Pythagorean Means", International Journal of Mathematics Trends and Technology (IJMTT) (ISSN: 2231 – 5373), 67(6), 97 – 118. http://www.ijmttjournal.org . doi:10.14445/22315373/IJMTT-V67I6P512.
71. Dhritikesh Chakrabarty (2018): "Understanding the Space of Research", *Biostatistics and Biometrics Open Access Journal*, (ISSN: 2573-2633), 4(5), 001 – 017: DOI: 10.19080/BBOAJ.2018.04.555642.

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 8, Issue 8 , August 2021

72. Dhritikesh Chakrabarty (2018): “Statistics and Bioscience: Association in Research”, *Significances of Bioengineering & Biosciences*, (ISSN 2637-8078), 2(5), 001 – 007: SBB.000546.2018. DOI: 10.31031/SBB.2018.02.000546.
73. Dhritikesh Chakrabarty (2019): “Association of Statistics with Biostatistics Research”, *Biometrics & Biostatistics International Journal*, 8(3), 104 – 109. DOI: 10.15406/bbij.2019.08.00279 .

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Dr. Dhritikesh Chakrabarty is also an awardee of the Post Doctoral Research Award by the University Grants Commission for the period 2002─05.

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