ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 1, January 2024

Asymmetric Geometric Linnik Distributions and Processes

Mariamma Antony

Associate Professor, Department of Statistics, Little Flower College, Guruvayoor, Kerala, India

ABSTRACT: Asymmetric generalizations of geometric Linnik distribution are studied in this paper. A representation of the generalized asymmetric Linnik distribution is obtained. Type I generalized geometric asymmetric Linnik distribution is introduced. It is shown that this distribution arises as the limit distribution of the geometric sums of generalized asymmetric Linnik random variables. The stability property of type I generalized geometric asymmetric Linnik distribution is examined. Autoregressive models with type I generalized geometric asymmetric Linnik marginals are developed.

KEY WORDS: Asymmetric Linnik distribution, Autoregressive Process, Geometric Sums, Laplace Distribution, Stationarity.

I. INTRODUCTION

As a generalization of Laplace distribution, Linnik distributions are studied by many authors. Note that Linnik laws are more heavy tailed as compared to Laplace. But both Laplace and Linnik distributions are symmetric. So, both of them have the draw back of modeling skewed data sets, that are common in various fields such as Finance, environmental studies etc. As an extension of Linnik Laws to model skewed data sets, asymmetric Linnik Laws are developed in the literature.

In this paper, we introduce and study Type I generalized geometric asymmetric Linnik distributions. In Section2, a representation of generalized asymmetric Linnik distribution is obtained. Type I Generalized geometric asymmetric Linnik distribution is studied in Section 3 and it is shown that this laws arise as the limit law of geometric sums of asymmetric generalized Linnik laws. The random stability property of Type I generalized geometric asymmetric Linnik laws are also studied in this Section. Autoregressive models of type I generalized geometric asymmetric Linnik marginal distribution are introduced and studied in Section 4.

II. GENERALIZED ASYMMETRIC LINNIK DISTRIBUTION

In this Section, we obtain a representation for an asymmetric version of the generalized Linnik distribution. Consider the distribution with characteristic function

$$\phi(t) = \left(\frac{1}{1+\lambda|t|^{\alpha} - i\mu t}\right)^{\tau}, -\infty < \mu < \infty, \lambda, \tau \geq 0, 0 < \alpha \leq 2. \qquad (2.1)$$

We shall refer this distribution as the generalized asymmetric Linnik distribution and denote it by $\mathrm{GeAL}(\alpha, \lambda, \mu, \tau)$. Note that when μ=0, (2.1) reduces to Pakes generalized Linnik distribution, see [3]. Also, when α =2, τ =1, (2.1) reduces to asymmetric Laplace distribution of [2].

THEOREM 2.1: A $\mathrm{GeAL}(\alpha, \lambda, \mu, \tau)$ random variable X with characteristic function (2.1) admits the representation $X \underline{\underline{d}} \mu W + (\lambda W)^{1/\alpha} Z$ where Z is symmetric stable with characteristic function $\psi(t) = e^{-\lambda|t|^{\alpha}}$ and W is a Gamma random variable with probability density function $g(w) = \frac{1}{\Gamma(\tau)} w^{\tau-1} e^{-w}, w > 0, \tau > 0$ independent of Z.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 1, January 2024

PROOF: Conditioning on W we obtain the characteristic function $\phi(t)$ *of* $\mu W + (\lambda W)^{1/\alpha} Z$ *as*

$$\phi(t) = E\left(E\left(e^{it\left(\mu W + (\lambda W)^{1/\alpha} Z \right)} \Big/ W \right) \right)$$

$$= \int_0^\infty E\left(e^{it\left(\mu w + (\lambda w)^{1/\alpha} Z \right)} \right) g(w)dw$$

$$= \left[\frac{1}{1+\lambda|t|^\alpha - i\mu t} \right]^\tau .$$

Hence the Theorem.

III.GENERALIZED GEOMETRIC ASYMMETRIC LINNIK DISTRIBUTION

Since the distribution with characteristic function (2.1) is infinitely divisible, using the result of [1], we can define a geometrically infinitely divisible distribution with characteristic function $\psi(t)$ as $\phi(t) = \exp\left\{1 - \frac{1}{\psi(t)}\right\}$ where $\phi(t)$ is the characteristic function of an infinitely divisible distribution. The characteristic function (2.1) can be written as

$$\left(\frac{1}{1+\lambda|t|^\alpha - i\mu t} \right)^\tau = \exp\left\{ 1 - \frac{1}{\left[1+\tau \ln(1+\lambda|t|^\alpha - i\mu t) \right]^{-1}} \right\}$$

Hence $\psi(t) = \frac{1}{1+\tau \ln(1+\lambda|t|^\alpha - i\mu t)}$ is a characteristic function of a geometrically infinitely divisible distribution.

The distribution with characteristic function

$$\psi(t) = \frac{1}{1+\tau \ln(1+\lambda|t|^\alpha - i\mu t)}, -\infty < \mu < \infty, \lambda, \tau \geq 0, 0 < \alpha \leq 2 \qquad (3.1)$$

is called Type I generalized geometric asymmetric Linnik distribution with parameters $\mu, \sigma, \alpha, \tau$. When μ=0, (3.1) reduces to type I generalized geometric Linnik distribution, see [4]. For time series models with Type I generalized geometric Linnik marginals, see [5].

If X is a random variable with characteristic function (3.1), we represent it as $X \underset{=}{d} GeGAL_1(\alpha, \lambda, \mu, \tau)$. It may be noted that when $\tau = 1$ in (3.1) the corresponding distribution is the geometric version of asymmetric Linnik distribution and we call it as geometric asymmetric Linnik distribution and is denoted by $X \overset{d}{=} GAL(\alpha, \lambda, \mu)$. Now we consider the asymmetric behavior of the $GeGAL_1$ distribution.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 1, January 2024

THEOREM 3.1: The $GeGAL_1(\alpha,\lambda,\mu,\tau)$ distribution is the limit distribution of the geometric sums of $GeAL\left(\alpha,\lambda,\mu,\frac{\tau}{n}\right)$ random variables.

PROOF: Let $\phi(t)$ be the characteristic function of a $GeAL\left(\alpha,\lambda,\mu,\frac{\tau}{n}\right)$ random variable. Then

$$\phi(t) \quad = \quad \left(\frac{1}{1+\lambda|t|^{\alpha}-i\mu t}\right)^{\frac{\tau}{n}}.$$

Define

$$\Theta(t)=\frac{1}{\phi(t)}-1=\left(1+\lambda|t|^{\alpha}-i\mu t\right)^{\frac{\tau}{n}}-1.$$

Hence using Lemma 3.2 of [6],

$$\phi_n(t)=\frac{1}{1+p\Theta(t)}$$

where p > 1, is the characteristic function of geometric sum of random variables. By choosing p = n, we have

$$\phi_n(t)=\left[1+n\left[\left(1+\lambda|t|^{\alpha}-i\mu t\right)^{\tau/n}-1\right]\right]^{-1}.$$

So $\phi_n(t)$ is the characteristic function of geometric sum of $GeAL\left(\alpha,\lambda,\mu,\frac{\tau}{n}\right)$.

Consider

$$\lim_{n\to\infty}\phi_n(t)=\frac{1}{1+\lim_{n\to\infty} n\left[\left(1+\lambda|t|^{\alpha}-i\mu t\right)^{\tau/n}-1\right]}$$

$$=\frac{1}{1+\tau\ln(1+\lambda|t|^{\alpha}-i\mu t)}$$

which is the characteristic function of $GeGAL_1(\alpha,\lambda,\mu,\tau)$ random variables. Hence the theorem.

Now we prove stability property of $GeGAL_1(\alpha,\lambda,\mu,\tau)$ random variables with respect to geometric summation.

THEOREM 3.2: Let $\{X_n\}$ be a sequence of independent and identically distributed random variables and let N_p be geometric random variable with mean $\frac{1}{p}$. Further, assume that N_p is independent of X_i's. If $U_{N_p}=\sum_{i=1}^{N_p} X_i$, then the random variables U_{N_p} and X_i are identically distributed if X_i follows $GeGAL_1(\alpha,\lambda,\mu,\tau)$.

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 1, January 2024

PROOF: Let $\phi(t)$ and $\Theta(t)$ be the characteristic functions of X_i and U_{N_p} respectively. Then

$$\Theta(t) = \frac{p\phi(t)}{1-(1-p)\phi(t)}. \qquad (3.2)$$

Suppose $X_i \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$.
Then, by (3.2) we have

$$\Theta(t) = \frac{p}{p+\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)} = \frac{1}{1+\frac{\tau}{p}\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}.$$

Hence the theorem.

IV.AUTOREGRESSIVE MODELS WITH GENERALIZED GEOMETRIC ASYMMETRIC LINNIK MARGINAL DISTRIBUTION

Here we develop a time series model with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal distribution on the basis of geometric infinite divisibility property of the distribution.

THEOREM 4.1: Let $\{X_n, n\geq 1\}$ be defined as

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad \theta \\ X_{n-1}+\varepsilon_n & w.p. \quad 1-\theta \end{cases} \qquad (4.1)$$

where $0<\theta\leq 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. A necessary and sufficient condition that $\{X_n\}$ is a stationary process with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal is that $\{\varepsilon_n\}$ is distributed as $GeGAL_1(\alpha,\lambda,\mu,\theta\tau)$.

PROOF: Let $\phi_{X_n}(t)$ be the characteristic function of $\{X_n\}$. Then from (4.1), we get

$$\phi_{X_n}(t) = \theta\,\phi_{\varepsilon_n}(t) + (1-\theta)\,\phi_{X_{n-1}}(t)\phi_{\varepsilon_n}(t) \qquad (.4.2)$$

Assuming stationarity, we have

$$\phi_X(t) = \theta\,\phi_{\varepsilon}(t) + (1-\theta)\,\phi_X(t)\,\phi_{\varepsilon}(t).$$

Hence

$$\phi_{\varepsilon}(t) = \frac{\phi_X(t)}{\theta+(1-\theta)\phi_X(t)}. \qquad (4.3)$$

Suppose $X_n \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$.
Then

$$\phi_X(t) = \frac{1}{1+\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}.$$

Substituting this in (4.3) and simplifying we get,

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 1, January 2024

$$\phi_{\varepsilon}(t) = \frac{1}{1+\theta\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}.$$

Hence $\varepsilon_n \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\theta\tau)$.

Conversely, assume that $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(\alpha,\lambda,\mu,\theta\tau)$ random variables and $X_0 \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$. Then from (4.2), for n = 1 we have

$$\phi_{X_1}(t) = \frac{1}{1+\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}.$$

If $X_{n-1} \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$, then we get $X_n \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$..

Thus using inductive argument, $\{X_n\}$ is a stationary process with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal distribution. Hence the Theorem.

We call the process defined by (4.1) with $X_0 \underset{=}{d} GeGAL_1(\alpha,\lambda,\mu,\tau)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(\alpha,\lambda,\mu,\theta\tau)$ random variables as the first order autoregressive process with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal distribution .

From the Definition of the model (4.1), it is easily verified that

$$\phi_{X_n}(t) = \theta\phi_{\varepsilon_n}(t)\frac{1-(1-\theta)^n\phi_{\varepsilon_n}^n(t)}{1-(1-\theta)\phi_{\varepsilon_n}(t)} + (1-\theta)^n\phi_{X_0}(t)\phi_{\varepsilon_n}^n(t).$$

When $n\to\infty$, $\phi_{X_n}(t) = \theta\phi_{\varepsilon_n}(t)\dfrac{1}{1-(1-\theta)\phi_{\varepsilon_n}(t)}$.

Let X_0 is distributed arbitrary and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(\alpha,\lambda,\mu,\theta\tau)$ random variables. Then as $n\to\infty$

$$\phi_{X_n}(t) = \frac{1}{1+\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}.$$

Hence if X_0 is distributed arbitrary, then the autoregressive process is asymptotically stationary with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal distribution.

Now from the joint characteristic function of (X_n, X_{n+1}) of the process, it can be easily verified that the first order autoregressive process (4.1) with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ marginal distribution is not time reversible.

An autoregressive model of k^{th} order with $GeGAL_1(\alpha,\lambda,\mu,\tau)$ as marginal distribution can be defined as

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p_0 \\ X_{n-1} + \varepsilon_n & w.p. \quad p_1 \\ X_{n-2} + \varepsilon_n & w.p. \quad p_2 \\ . & \\ . & \\ . & \\ X_{n-k} + \varepsilon_n & w.p. \quad p_k \end{cases}$$

where $\sum_{i=0}^{k} p_i = 1$, $0 < p_i < 1$, $i = 1..k$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(\alpha, \lambda, \mu, \tau)$ random variables.

REFERENCES

[1]. L.B. Klebanov, G.M. Maniya, and I.A. Melamed, "A problem of Zolotarev and Analogs of Infinitely Divisible and Stable Distributions in a Scheme for Summing a Random Number of Random Variables", Probability Theory and Its Applications, vol. 29, 1984, pp.791-794.

[2]. T.J. Kozubowski and K. Podgorski, "Asymmetric Laplace Distributions", Mathematical Scientist , vol. 25, 2000, pp.37-46.

[3]. V.S. Lekshmi,and K.K. Jose, "Autoregressive process with Pakes and geometric Pakes generalized Linnik marginals", Statistics and Probability Letters, vol. 76, 2006, pp.318-326.

[4]. A. Mariamma, "Generalized Geometric Linnik Distribution", Asian Journal of Statistical Sciences, vol. 2, 2022, pp.105-110.

[5] .A. Mariamma, "Time Series Models With Generalized Geometric Linnik Marginals", Asian Journal of Statistical Sciences, vol. 3, 2023, pp.57-60.

[6]. R.N. Pillai, "Harmonic Mixtures and Geometric Infinite Divisibility", Journal of the Indian Statistical Association, vol. 28, 1990, pp.87-98.