ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 10, October 2024

On finding Integer Solutions to Binary Quintic Equation $x^2 - x y^2 = y^4 + y^5$

J.Shanthi*, M.A.Gopalan

Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy- 620 002, Tamil Nadu, India.
Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

ABSTRACT: The process of obtaining many integer solutions to non-homogeneous polynomial equation of degree five with two unknowns given by $x^2 - x y^2 = y^4 + y^5$ is illustrated. A few relations between the solutions are presented. A procedure for obtaining second order Ramanujan numbers through integer solutions of the given binary quintic equation is illustrated.

I. INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics.The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Quintic Diophantine equations with multi variables [1-9]. It seems that much work has not been done regarding polynomial Diophantine equations of degree five with two unknowns. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree five with two unknowns given by $x^2 - x y^2 = y^4 + y^5$. A few relations between the solutions are presented. A procedure for obtaining second order Ramanujan numbers through integer solutions of the given binary quintic equation is illustrated.

II. METHOD OF ANALYSIS

The non-homogeneous quintic equation with two unknowns under consideration is

$$x^2 - x y^2 = y^4 + y^5 \qquad (1)$$

To start with ,it is observed by scrutiny that (1) is satisfied by

$$x = k(k^2 - k - 1)^2 , y = (k^2 - k - 1) , k \neq 0$$

However , there are other choices of integer solutions to (1) and the process of obtaining the same is illustrated below :

Treating (1) as a quadratic in x and solving for the same ,we have

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 10, October 2024

$$x=\frac{y^2[1\pm\sqrt{4y+5}]}{2} \qquad (2)$$

Let

$$\alpha^2=4y+5 \qquad (3)$$

which , after some algebra , is satisfied by

$$y_0=s(s+1)-1\,,\alpha_0=2s+1 \qquad (4)$$

Assume the second solution to (3) as

$$\alpha_1=h-\alpha_0\,,y_1=h+y_0 \qquad (5)$$

where h is an unknown to be determined. Substituting (5) in (3) and simplifying , we have

$$h=2\alpha_0+4$$

and in view of (5) ,it is seen that

$$\alpha_1=\alpha_0+4, y_1=y_0+2\alpha_0+4$$

The repetition of the above process leads to the general solution to (3) as

$$\alpha_n=\alpha_0+4n=2s+1+4n,$$

$$y_n=y_0+2n\alpha_0+4n^2=(s+2n)\,(s+2n+1)-1 \qquad (6)$$

From (2) ,we get

$$x_n=\frac{y_n^2[1\pm\alpha_n]}{2}$$

$$=(2n+s+1)\,y_n^2\;,-(2n+s)\,y_n^2$$

Thus , we have two sets of integer solutions to (1) represented by

Set 1

$$x_n=x_n(s)=(s+2n+1)[(s+2n+1)(s+2n)-1]^2\;,$$

$$y_n=y_n(s)=(s+2n)\,(s+2n+1)-1.$$

Set 2

$$x_n=x_n(s)=-(s+2n)[(s+2n+1)\,(s+2n)-1]^2\;,$$

$$y_n=y_n(s)=(s+2n)\,(s+2n+1)-1.$$

A. Considering Set 1 ,the following relations are observed :

(i) $y_{n+2}(s)-2y_{n+1}(s)+y_n(s)=8, n=0,1,2,...$

(ii) $y_n(s)+1=2t_{3,s+2n}$

(iii) $y_n^3(s)-x_n(s)+2y_n^2(s)$ is a perfect square

(iv) $\dfrac{x_n(s)}{y_n^2(s)}$ is a perfect square when $s=k^2n^2+2(k-1)n$

(v) $\dfrac{x_n(s)}{y_n^2(s)}=P_n^5$ when $s=n^3+(n-1)^2-2$

(vi) $y_n(s+2)-2y_n(s+1)+y_n(s)=2$

(vii) $y_n^2(s)\,[y_{n+2}(s)-y_{n+1}(s)-10]=4x_n(s)$

(viii) $y_n^2(s)\,[y_{n+2}(s)-y_n(s)-12]=8x_n(s)$

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 10, October 2024

(ix) $\frac{x_n(s)}{y_n^2(s)} - s$ is a perfect square when $n = 4\, t_{3,N}$

(x) $\left(\frac{x_n(s)}{y_n^2(s)} - s\right)^2 - 1 = 8\, t_{3,n}$

(xi) $\frac{x_n(s)}{y_n^2(s)} - 1 = t_{2k+4,n}$ when $s = (k+1)n^2 - (k+2)n, k \geq 1$

(xii) $\frac{x_n(s)}{y_n^2(s)} - 1 = t_{2k+3,n}$ when $s = (2k+1)n^2 - (2k+1)n, k \geq 0$

(xiii) $x_n^2(s) - y_n^2(s)[y_n^3(s) - x_n(s) + 2y_n^2(s)] = 2x_n(s)\, y_n^2(s) - y_n^4(s)$

(xiv) From the integer solutions to (1) given by Set 1,one may generate Second order Ramanujan numbers as shown below :
Illustration:

$$y_3(1) + 1 = 56 = 1*56 = 2*28 = 4*14 = 7*8$$
$$= \quad F_1 \qquad F_2 \qquad F_3 \qquad F_4$$
$$F_1 = F_2 \Rightarrow (56+1)^2 + (28-2)^2 = (56-1)^2 + (28+2)^2$$
$$= 57^2 + 26^2 = 55^2 + 30^2 = 3925$$
$$F_1 = F_3 \Rightarrow (56+1)^2 + (14-4)^2 = (56-1)^2 + (14+4)^2$$
$$= 57^2 + 10^2 = 55^2 + 18^2 = 3349$$
$$F_1 = F_4 \Rightarrow (56+1)^2 + (8-7)^2 = (56-1)^2 + (8+7)^2$$
$$= 57^2 + 1^2 = 55^2 + 15^2 = 3250$$
$$F_2 = F_4 \Rightarrow (28+2)^2 + (8-7)^2 = (28-2)^2 + (8+7)^2$$
$$= 30^2 + 1^2 = 26^2 + 15^2 = 901$$
$$F_3 = F_4 \Rightarrow (14+4)^2 + (8-7)^2 = (14-4)^2 + (8+7)^2$$
$$= 18^2 + 1^2 = 10^2 + 15^2 = 325$$

Thus , 3925 ,3349 ,3250 ,901 ,325 represent second order Ramanujan numbers .
A similar observation may be performed by considering the solutions in Set 2.

III. CONCLUSION

The polynomial equation of degree five with two unknowns has been studied to obtain non-zero integer solutions .The process of eliminating the square-root will be beneficial for the researchers. As quintic equations are plenty , one may attempt to determine the solutions in integers for other choices of quintic diophantine equations.

REFERENCES

[1] R. Anbuselvi, R. Nandhini, Integer solutions of non-homogeneous quintic equation with five unknowns $4(x^4 - y^4) = 50(z^2 - w^2)m^3$ International Journal of Innovative Research in Engineering & Multidisciplinary Physical Sciences,7(1), 103-107, 2019 https://www.ijirmps.org/papers/2019/1/492.pdf

[2] N.Thiruniraiselvi, M.A.Gopalan., The Non-Homogeneous Quintic Equation with Six Unknowns $(x^4 - y^4) = 109(z + w)P^3Q$, International Journal for Research in Applied Science & Engineering Technology ,7(VI), 2527-2530, 2019. https://www.ijraset.com/fileserve.php?FID=23937

[3] A.Vijayasankar, Sharadha Kumar, M.A.Gopalan. On The Non-Homogeneous Quintic Equation with Five Unknowns $3(x+y)(x^3 - y^3) = 7(z^2 - w^2)p^3$, 10(8) ,44-49 , 2020. https://www.researchinventy.com/papers/v10i8/F10084449.pdf

[4] S.Vidhyalakshmi, T. Mahalakshmi, G. Dhanalakshmi, M.A. Gopalan, On The Non-Homogeneous Quintic Equation with Five Unknowns $(x^4 - y^4) = 13P^3(z^2 - w^2)$, IAR Journal of Engineering and Technology,1(1), 43-49,2020. https://www.iarconsortium.org/iarjet/33/147/on-the-non-homogeneous-quintic-equation-with-five-unknowns-x4-y4-13p3-z2-w2-35/

[5] A.Vijayasankar, Sharadha Kumar, M.A.Gopalan ,ON NON-HOMOGENEOUS QUINARY QUINTIC EQUATION $(x^4 - y^4) = 125(z^2 - w^2)p^3$, South East Asian J. of Mathematics and Mathematical Sciences,18(1), 27-34, 2022. https://rsmams.org/download/articles/2_18_1_197960429_ON%20NON%20HOMOGENEOUS%20QUINARY%20QUINTIC%20EQUATION.pdf

[6] J.Shanthi , M.A.Gopalan , On the Non-homogeneous Quinary Quintic Equation $x^4 + y^4 - (x+y)w^3 = 14z^2T^3$, International Research Journal of Education and Technology , 05 (08) ,238-245 , 2023. https://www.irjweb.com/On%20The%20%20Non%20homogeneous%20Quinary%20%20%20Quintic%20Equation.pdf

[7] T. Mahalakshmi , E. Shalini, On Finding Integer Solutions to The Non-Homogeneous Ternary Quintic Diophantine Equation $3(x^2 + y^2) - 5xy = 15z^5$, International Research Journal of Education and Technology ,05(03) ,472-478, 2023. https://www.irjweb.com/ON%20FINDIND%20INTEGER%20SOLUTIONS%20TO%20THE%20NONHOMOGENEOUS%20TERNARY%20QUINTIC%20DIOPHANTINE%20EQUATION.pdf

[8] J.Shanthi , M.A.Gopalan , Delineation of Integer Solutions to Non-Homogeneous Quinary Quintic Diophantine Equation $(x^3 - y^3) - (x^2 + y^2) + (z^3 - w^3) = 2 + 87T^5$, International Journal of Research Publication and Reviews, 4(9), 1454-1457 , 2023. https://ijrpr.com/uploads/V4ISSUE9/IJRPR17233.pdf

[9] J.Shanthi , M.A.Gopalan , On the Ternary Non-homogeneous Quintic Equation $x^2 + 5y^2 = 2z^5$, International Journal of Research Publication and Reviews, 4(8), 329-332, 2023. https://ijrpr.com/uploads/V4ISSUE8/IJRPR16050.pdf