ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
Vol. 11, Issue 9, September 2024

Solving Fredholm Integral Equations of the Second Kind by Using Midpoint Quadrature Method

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ABSTRACT: In this paper, we propose an approach to find approximate solution to the linear Fredholm integral equation of the second kind by midpoint quadrature method. It is one of the simplest methods of numerical integration and is based on the idea of evaluating the function at the midpoint of subintervals to estimate the area under the curve. The results reveal that the method is accurate and easy to implement.

I. INTRODUCTION

Integral equations, a fundamental concept in mathematics, encompass the relationship where an unknown function appears under an integral sign. Integral equations are often encountered in fields such as physics, engineering, and applied mathematics. They are closely related to differential equations and can describe a wide variety of physical phenomena [1], [2]. There are several studies of Adomian decomposition method, convergence and accuracy of Adomian's decomposition method for the solution of Lorenz equations is studied in [3]. Solving Riccati differential equation using Adomian's decomposition method is given in [4]. M. Hasan studied approximate solution of nonlinear integral equations of the second kind by using Homotopy Perturbation Method [5]. Many initial and boundary value problems associated with ordinary differential equation and partial differential equation can be transformed into problems of solving some approximate integral equations ([6] and [7]). Recently, the applications of Homotopy perturbation method theory have appeared in the works of many scientists [8], [9], [10], [11], [12], [13], which shows that the method has become a powerful mathematical tool [14].

In this article we have applied the midpoint quadrature method used by using the MATLAB algorithm. By applying this algorithm to different examples, including finding the approximate solution and then comparing it to the exact solution and finding out the amount of error between the approximate solution and the exact solution. The main objective of this work is to use the midpoint quadrature method in solving the Fredholm integral equation of the second kind using MATLAB. Consider the linear Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)dt \qquad (i)$$

It can be easily observed that the unknown function $y(x)$ appears under the integral sign. It is to be noted here that both $K(x,t)$ the kernel function, which is known and the function $f(x)$ in equation (i) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $y(x)$ that will satisfy equation (i) using a number of solution techniques.

II. MIDPOINT QUADRATURE METHOD

The Midpoint Quadrature Method is a numerical technique used to approximate the integral of a function, particularly useful in solving integral equations. It is a simple, yet effective method that approximates the integral by evaluating the function at the midpoint of subintervals. From equation (i), the linear Fredholm integral equation of second kind is

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)dt$$

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International Journal of AdvancedResearch in Science, Engineering and Technology
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Let x_i be the quadrature points then (i) can be written as

$$y\left(x_{i+\left(\frac{1}{2}\right)}\right) = f\left(x_{i+\left(\frac{1}{2}\right)}\right) + \int_a^b K\left(x_{i+\left(\frac{1}{2}\right)}, t\right) y(t)dt$$

$$\text{or, } y\left(x_{i+\left(\frac{1}{2}\right)}\right) = f\left(x_{i+\left(\frac{1}{2}\right)}\right) + \sum_{j=0}^{N-1} hK\left(x_{i+\left(\frac{1}{2}\right)}, x_{j+\left(\frac{1}{2}\right)}\right) y\left(x_{j+\left(\frac{1}{2}\right)}\right)$$

$$\text{or, } y\left(x_{i+\left(\frac{1}{2}\right)}\right) = f\left(x_{i+\left(\frac{1}{2}\right)}\right) + \sum_{j=0}^{N-1} hK\left(x_{i+\left(\frac{1}{2}\right)}, x_{j+\left(\frac{1}{2}\right)}\right) y\left(x_{j+\left(\frac{1}{2}\right)}\right)$$

For $i\ = 0, 1, 2$ we can write

$$y\left(x_{\left(\frac{1}{2}\right)}\right) = f\left(x_{\left(\frac{1}{2}\right)}\right) + h\left(K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) y\left(x_{\left(\frac{1}{2}\right)}\right) + K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) y\left(x_{\left(\frac{3}{2}\right)}\right) + \cdots \ldots \ldots \ldots . + K\left(x_{\left(\frac{1}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) y\left(x_{N-\left(\frac{1}{2}\right)}\right)\right)$$

$$\text{or, } y\left(x_{\left(\frac{3}{2}\right)}\right) = f\left(x_{\left(\frac{3}{2}\right)}\right) + h\left(K\left(x_{\left(\frac{3}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) y\left(x_{\left(\frac{1}{2}\right)}\right) + K\left(x_{\left(\frac{3}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) y\left(x_{\left(\frac{3}{2}\right)}\right) + \cdots \ldots \ldots \ldots . + K\left(x_{\left(\frac{3}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) y\left(x_{N-\left(\frac{1}{2}\right)}\right)\right)$$

$$\text{or, } y\left(x_{\left(\frac{5}{2}\right)}\right) = f\left(x_{\left(\frac{5}{2}\right)}\right) + h\left(K\left(x_{\left(\frac{5}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) y\left(x_{\left(\frac{1}{2}\right)}\right) + K\left(x_{\left(\frac{5}{2}\right)}, x_{\left(\frac{5}{2}\right)}\right) y\left(x_{\left(\frac{5}{2}\right)}\right) + \cdots \ldots \ldots \ldots . + K\left(x_{\left(\frac{5}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) y\left(x_{N-\left(\frac{1}{2}\right)}\right)\right)$$

and so on.
Therefore, the system of linear equations becomes:

$$\begin{pmatrix} y\left(x_{\left(\frac{1}{2}\right)}\right) \\ y\left(x_{\left(\frac{3}{2}\right)}\right) \\ y\left(x_{\left(\frac{5}{2}\right)}\right) \\ \ldots \ldots \\ \ldots \ldots \\ y\left(x_{N-\left(\frac{1}{2}\right)}\right) \end{pmatrix} = \begin{pmatrix} K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) & K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) \ldots . K\left(x_{\left(\frac{1}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) \\ K\left(x_{\left(\frac{3}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) & K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) \ldots . K\left(x_{\left(\frac{1}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) \\ K\left(x_{\left(\frac{5}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) & K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) \ldots . K\left(x_{\left(\frac{1}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ K\left(x_{N-\left(\frac{1}{2}\right)}, x_{\left(\frac{1}{2}\right)}\right) & K\left(x_{\left(\frac{1}{2}\right)}, x_{\left(\frac{3}{2}\right)}\right) \ldots . K\left(x_{N-\left(\frac{1}{2}\right)}, x_{N-\left(\frac{1}{2}\right)}\right) \end{pmatrix} \begin{pmatrix} y\left(x_{\left(\frac{1}{2}\right)}\right) \\ y\left(x_{\left(\frac{3}{2}\right)}\right) \\ y\left(x_{\left(\frac{5}{2}\right)}\right) \\ \ldots \ldots \\ \ldots \ldots \\ y\left(x_{N-\left(\frac{1}{2}\right)}\right) \end{pmatrix} + \begin{pmatrix} f\left(x_{\left(\frac{1}{2}\right)}\right) \\ f\left(x_{\left(\frac{3}{2}\right)}\right) \\ f\left(x_{\left(\frac{5}{2}\right)}\right) \\ \ldots \ldots \\ \ldots \ldots \\ f\left(x_{N-\left(\frac{1}{2}\right)}\right) \end{pmatrix}$$

$$or, \quad Y = AY + F$$
$$or, \quad (I - A)\, Y = F$$
$$or, \quad Y = \ (I - A)^{-1} F$$

which is the system of linear Fredholm integral equation of the second kind of equation (i).

III. HIGHER ORDER QUADRATURE METHOD WITH WEIGHTS

Higher-order quadrature methods with weights, also known as Gaussian Quadrature or Weighted Quadrature, are numerical integration techniques that use strategically chosen points (nodes) and corresponding weights to approximate definite integrals more accurately. These methods are particularly effective because they are tailored to approximate integrals of polynomials of higher degrees with fewer points compared to basic methods like the Midpoint or Trapezoidal rule. Given that

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$$f(x_{l,i}) = \sum_{j=1}^{N} \int_{\Omega_j} K(x_{l,i} - y) f(y)\, dy + g(x_{l,i}) \qquad (ii)$$

Now

$$\int_{\Omega_j} K(x_{l,i} - y) f(y)\, dy = \sum_{k=1}^{M} w_{k,j} K(x_{l,i} - x_{k,j}) f(x_{k,j})$$

From (ii) we have

$$f(x_{l,i}) = \sum_{j=1}^{N} \sum_{k=1}^{M} w_{k,j} K(x_{l,i} - x_{k,j}) f(x_{k,j}) + g(x_{l,i})$$

$= \sum_{j=1}^{N} \{w_{1,j} K(x_{l,i} - x_{1,j}) f(x_{1,j}) + w_{2,j} K(x_{l,i} - x_{2,j}) f(x_{2,j}) + w_{3,j} K(x_{l,i} - x_{3,j}) f(x_{3,j}) + w_{4,j} K(x_{l,i} - x_{4,j}) f(x_{4,j}) + w_{5,j} K(x_{l,i} - x_{5,j}) f(x_{5,j}) + \cdots \ldots \ldots\} + g(x_{l,i})$

$= \{w_{1,1} K(x_{l,i} - x_{1,1}) f(x_{1,1}) + w_{2,1} K(x_{l,i} - x_{2,1}) f(x_{2,1}) + w_{3,1} K(x_{l,i} - x_{3,1}) f(x_{3,1}) + w_{4,1} K(x_{l,i} - x_{4,1}) f(x_{4,1}) + w_{5,1} K(x_{l,i} - x_{5,1}) f(x_{5,1}) + \cdots$

$\ldots\ldots + w_{1,2} K(x_{l,i} - x_{1,2}) f(x_{1,2}) + w_{2,2} K(x_{l,i} - x_{2,2}) f(x_{2,2}) + w_{3,2} K(x_{l,i} - x_{3,2}) f(x_{3,2}) + w_{4,2} K(x_{l,i} - x_{4,2}) f(x_{4,2}) + w_{5,2} K(x_{l,i} - x_{5,2}) f(x_{5,2}) + \cdots$

$\ldots\ldots + w_{1,3} K(x_{l,i} - x_{1,3}) f(x_{1,3}) + w_{2,3} K(x_{l,i} - x_{2,3}) f(x_{2,3}) + w_{3,3} K(x_{l,i} - x_{3,3}) f(x_{3,3}) + w_{4,3} K(x_{l,i} - x_{4,3}) f(x_{4,3}) + w_{5,3} K(x_{l,i} - x_{5,3}) f(x_{5,3}) + \cdots \ldots\ldots\} + g(x_{l,i})$

$= w_{1,1} K(x_{l,i} - x_{1,1}) f(x_{1,1}) + w_{1,2} K(x_{l,i} - x_{1,2}) f(x_{1,2}) + \cdots .. + w_{1,M} K(x_{l,i} - x_{1,M}) f(x_{1,M})$
$+ w_{2,1} K(x_{l,i} - x_{2,1}) f(x_{2,1}) + w_{2,2} K(x_{l,i} - x_{2,2}) f(x_{2,2}) + \cdots .. + w_{2,M} K(x_{l,i} - x_{2,M}) f(x_{2,M})$
$+ w_{3,1} K(x_{l,i} - x_{3,1}) f(x_{3,1}) + w_{3,2} K(x_{l,i} - x_{3,2}) f(x_{3,2}) + \cdots .. + w_{3,M} K(x_{l,i} - x_{3,M}) f(x_{3,M})$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$+ w_{N,1} K(x_{l,i} - x_{N,1}) f(x_{N,1}) + w_{1,2} K(x_{l,i} - x_{N,2}) f(x_{N,2}) + \cdots .. + w_{1,M} K(x_{l,i} - x_{N,M}) f(x_{N,M})$
$+ g(x_{l,i})$

$$f(x_{l,i}) = \begin{pmatrix} w_{1,1} K(x_{l,i} - x_{1,1}) & w_{1,2} K(x_{l,i} - x_{1,2}) \ldots\ldots & w_{1,M} K(x_{l,i} - x_{1,M}) \\ w_{2,1} K(x_{l,i} - x_{2,1}) & w_{2,2} K(x_{l,i} - x_{2,2}) \ldots\ldots & w_{2,M} K(x_{l,i} - x_{2,M}) \\ w_{3,1} K(x_{l,i} - x_{3,1}) & w_{3,2} K(x_{l,i} - x_{3,2}) \ldots\ldots & w_{3,M} K(x_{l,i} - x_{3,M}) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ w_{N,1} K(x_{l,i} - x_{N,1}) & w_{N,2} K(x_{l,i} - x_{N,2}) \ldots\ldots\ldots & w_{N,M} K(x_{l,i} - x_{N,M}) \end{pmatrix} \begin{pmatrix} f(x_{1,1}) & f(x_{1,2}) \ldots\ldots f(x_{1,M}) \\ f(x_{2,1}) & f(x_{2,2}) \ldots\ldots f(x_{2,M}) \\ f(x_{3,1}) & f(x_{3,2}) \ldots\ldots f(x_{3,M}) \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ f(x_{N,1}) & f(x_{N,2}) \ldots\ldots f(x_{N,M}) \end{pmatrix} + g(x_{l,i})$$

IV. NUMERICAL IMPLEMENTATIONS

The Midpoint Quadrature Method is a straightforward numerical technique to approximate definite integrals. In this section, we consider some examples which is closely related to engineering and to applied the method described in the previous section.

Example 1. Consider the linear Fredholm integral equation of the second kind

$$y(x) = \frac{3}{2} e^{x} - \frac{1}{2} x e^{x} - \frac{1}{2} + \frac{1}{2} \int_{0}^{1} t\, y(t) dt \qquad (iii)$$

where the exact solution is

$$y(x) = \frac{3}{2} e^{x} - \frac{1}{2} x e^{x} - \frac{1}{3} e + 1$$

By applying the Midpoint quadrature method, we have

Table 1. Exact and approximate solution to the integral equation (iii)

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Node	*Approximate values*	*Exact values*	*Error*
0.02381	1.6177	1.5937	0.0149250
0.07143	1.6664	1.6445	0.0133270
0.11905	1.7163	1.6964	0.0117460
0.16667	1.7673	1.7495	0.0101840
0.21429	1.8195	1.8039	0.0086432
0.26190	1.8727	1.8594	0.0071242
0.30952	1.9270	1.9162	0.0056290
0.35714	1.9824	1.9742	0.0041596
0.40476	2.0388	2.0333	0.0027177
0.45238	2.0962	2.0935	0.0013054
0.50000	2.1546	2.1548	7.5347e-05
0.54762	2.2140	2.2171	0.0014222
0.59524	2.2742	2.2804	0.0027328
0.64286	2.3353	2.3447	0.0040046
0.69048	2.3971	2.4097	0.0052349
0.73810	2.4596	2.4755	0.0064208
0.78571	2.5228	2.5420	0.0075591
0.83333	2.5865	2.6090	0.0086465
0.88095	2.6506	2.6765	0.0096791
0.92857	2.7150	2.7443	0.0106530
0.97619	2.7797	2.8122	0.0115640

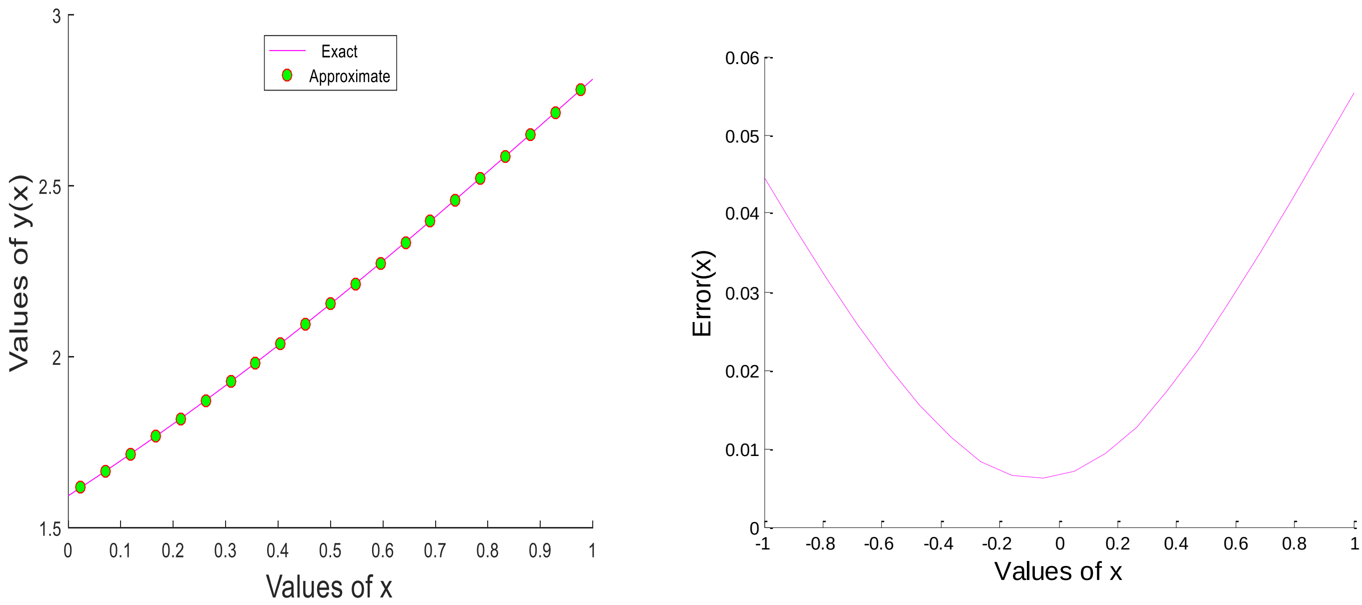

Fig. 1. Solutions and errors to the integral equation (iii)

Example 2. Consider the linear Fredholm integral equation

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International Journal of AdvancedResearch in Science, Engineering and Technology
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$$u(x) = e^x - \int_{-1}^{1} |x - t|\, u(t)dt \qquad (iv)$$

The exact solution to this equation is $u(x) = \frac{1}{2}xe^x + c_1e^x + c_2e^{-x}$, where $c_1 = c_2 + (e^2+1)^{-1}$ and $c_2 = (e^4 + 6e^2 + 1)/8(e^2+1)$. By applying the Midpoint quadrature method, we have

Table 2. Exact and approximate solution to the integral equation (iv)

Node	*Exact values*	*Approximate values*	*Error*
-0.9500	4.4553	4.2571	0.0445
-0.8500	4.1177	3.9606	0.0381
-0.7500	3.8302	3.7080	0.0319
-0.6500	3.5903	3.4971	0.0260
-0.5500	3.3958	3.3265	0.0204
-0.4500	3.2452	3.1949	0.0155
-0.3500	3.1375	3.1016	0.0114
-0.2500	3.0723	3.0464	0.0084
-0.1500	3.0496	3.0295	0.0066
-0.0500	3.0703	3.0514	0.0062
0.0500	3.1356	3.1134	0.0071
0.1500	3.2473	3.2171	0.0093
0.2500	3.4081	3.3645	0.0128
0.3500	3.6210	3.5585	0.0173
0.4500	3.8902	3.8022	0.0226
0.5500	4.2204	4.0997	0.0286
0.6500	4.6172	4.4555	0.0350
0.7500	5.0872	4.8750	0.0417
0.8500	5.6380	5.3645	0.0485
0.9500	6.2786	5.9310	0.0554

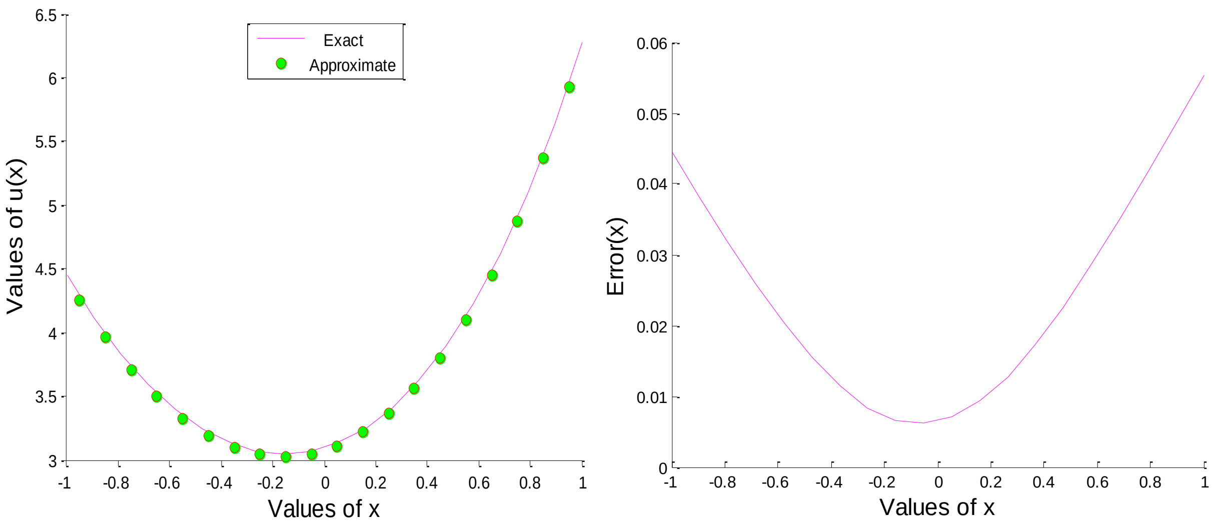

Fig. 2. Solutions and errors to the integral equation (iv)

Example 3. Consider the Fredholm integral equation of the second kind

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$$y(x) = \sqrt{x} - \int_0^1 \sqrt{xy}\, y(t)dt \qquad (v)$$

The exact solution to this equation $y(x) = (2/3)\sqrt{x}$. By applying the Midpoint quadrature method, we have

Table 3. Exact and approximate solution to the integral equation (v)

Node	*Exact values*	*Approximate values*	*Absolute Error*
0.0238	0	0.1029	0.1029
0.0714	0.1491	0.1782	0.0291
0.1190	0.2108	0.2300	0.0192
0.1667	0.2582	0.2722	0.0140
0.2143	0.2981	0.3086	0.0105
0.2619	0.3333	0.3412	0.0078
0.3095	0.3651	0.3709	0.0058
0.3571	0.3944	0.3984	0.0040
0.4048	0.4216	0.4241	0.0025
0.4524	0.4472	0.4484	0.0012
0.5000	0.4714	0.4714	0
0.5476	0.4944	0.4933	0.0011
0.5952	0.5164	0.5143	0.0021
0.6429	0.5375	0.5345	0.0030
0.6905	0.5578	0.5540	0.0038
0.7381	0.5774	0.5727	0.0046
0.7857	0.5963	0.5909	0.0053
0.8333	0.6146	0.6086	0.0061
0.8810	0.6325	0.6257	0.0067
0.9286	0.6498	0.6424	0.0074
0.9762	0.6667	0.6587	0.0080

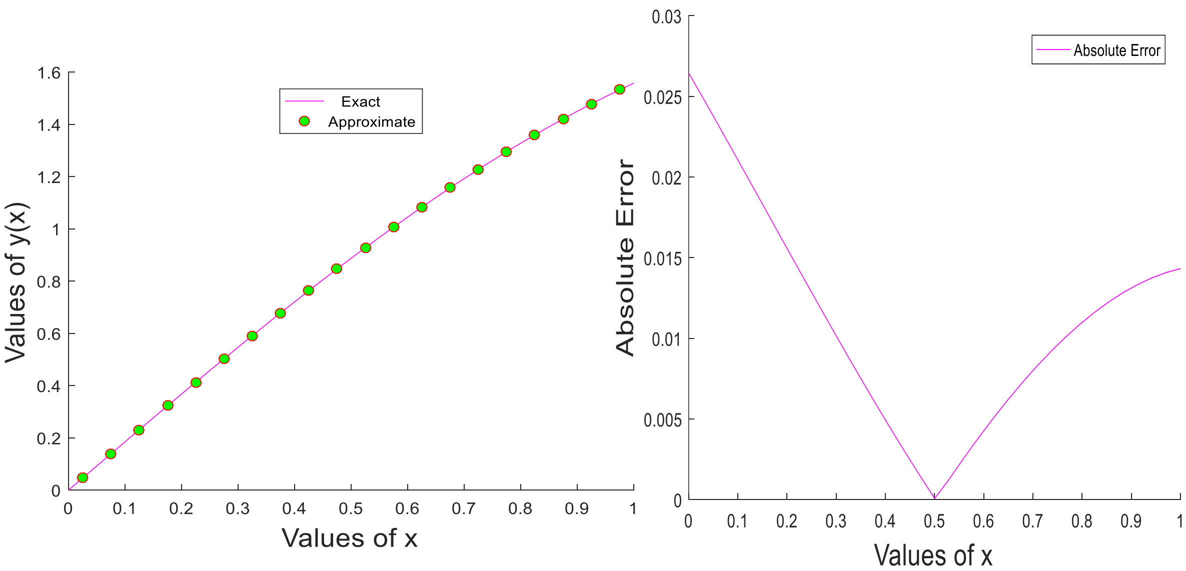

Fig. 3. Solutions and errors to the integral equation (v)

V. CONCLUSION

ISSN: 2350-0328
International Journal of AdvancedResearch in Science, Engineering and Technology
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Integral equations are a powerful mathematical tool with diverse applications across many fields. Ongoing research continues to improve numerical methods, and explore new applications. This paper presents a technique to find the approximate result of a linear Fredholm integral equation by midpoint quadrature method. The estimated solutions obtained by the midpoint quadrature method are compared with exact solutions. It can be concluded that the midpoint quadrature method is effective and accuracy. The numerical results demonstrations that the proposed method is well suited for the numerical solution of such kind problems

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