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Numerical Modeling of Unsteady Heat Transfer in a Non-Homogeneous Axisymmetric Body

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ABSTRACT: The distribution of the temperature field of elements of three-dimensional structures is examined in the article and the influence of non-homogeneities on the temperature field in three-dimensional axisymmetric structures is studied. Based on the variational formulation of the problem, a finite element model for solving axisymmetric problems of heat distribution in three-dimensional bodies was built based on the finite element method; a computational algorithm of the solution and the software were implemented. Shape functions corresponding to linear triangular elements are used in the finite element solution to the problem. They ensure its continuity at the boundary with neighboring elements since this distribution is linear along any side of the triangle and with the same change in value at the nodes, the same changes will occur along the entire inner boundary. A discrete finite element mesh model is a set of parameters that consist of the total number of finite elements and mesh nodes, arrays of their node coordinates, and node numbers of finite elements. To verify the reliability of the results obtained, a computational experiment was conducted, related to the study of the effect of the increase in the number of finite elements on the convergence of solutions. Analysis of the experimental results confirms the numerical convergence of temperature values as the finite element mesh is refined. At the initial stage of the study, the influence of temperature distribution for homogeneous structural elements was studied.

I. INTRODUCTION

Many significant three-dimensional physical problems can be simplified by using two-dimensional elements. For instance, the problem of radial heat flux through concentric cylinders with varying thermal conductivities illustrates this concept. In a long cylinder, heat flow occurs in both radial and axial directions, and if the boundary conditions are independent of the azimuthal angle θ, the heat flux remains unaffected by it. Similarly, the plane flow of water to a well is another example of an axially symmetric problem, where the flow characteristics are independent of the azimuthal angle θ. Such problems frequently appear in applications, including heat transfer and hydrodynamics, with water flow through porous media being a notable example [1]. When applying the finite element method (FEM), the primary adaptation lies in the dimensionality of the elements. Two-dimensional symmetric problems are reduced to one-dimensional ones, while three-dimensional axisymmetric problems are addressed using two-dimensional elements [2-3].

The Finite Element Method (FEM) [4] is a numerical procedure for solving problems formulated as differential equations or variational principles. Unlike classical Ritz and Galerkin methods, FEM employs an approximating function that is a linear combination of continuous, piecewise-smooth finite functions. These finite functions are non-zero only within specific intervals. In FEM, such intervals correspond to finite elements obtained by partitioning the volume V. The numerical methods of linear algebra [5] include techniques for solving systems of linear algebraic equations, matrix inversion, determinant calculation, and finding eigenvalues and eigenvectors of matrices. Methods for solving systems of linear algebraic equations are divided into two groups. The first group includes direct or exact methods, such as Cramer's rule, Gaussian elimination, and the sweep (or Progonka) method. Cramer's rule is rarely used in computing as it requires significantly more arithmetic operations than Gaussian elimination. Gaussian elimination, on the other hand, is effective for solving systems of up to the 10^3 order on computers and is employed in iterative methods for systems of up to the 10^6 order.

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II. MATERIAL AND METHODS

The axisymmetric unsteady heat transfer problem in a cylindrical coordinate system is described by the following differential equation [1]:

$$K_{rr}\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}K_{rr}\frac{\partial T}{\partial r}+\frac{K_{\theta\theta}}{r^2}\frac{\partial^2 T}{\partial \theta^2}+K_{zz}\frac{\partial^2 T}{\partial z^2}+Q=\rho c\frac{\partial T}{\partial t} \tag{1}$$

where $T=T(r,z,\theta,t)$ is the temperature; $K_{rr},K_{\theta\theta},K_{zz}$ are the thermal conductivity coefficients in the respective directions; $Q=Q(r,z,\theta,t)$ - the power of heat sources within the body; ρ - material density; c – heat capacity of the material; r - the distance from the symmetry axis to the center of the element; θ - - azimuthal angle. If a three-dimensional body has geometric symmetry about the *z*-axis, it is called an axisymmetric body (Fig 1). Moreover, if the physical quantity under study is independent of the azimuthal angle θ, then the differential equation (1) takes the following form:

$$K_{rr}\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}K_{rr}\frac{\partial T}{\partial r}+K_{zz}\frac{\partial^2 T}{\partial z^2}+Q=\rho c\frac{\partial T}{\partial t}, \tag{2}$$

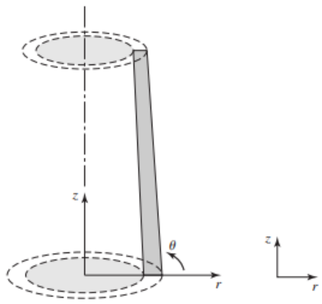

Fig 1. Appearance of an axisymmetric body.

To solve the problem, its variational form is considered, which allows the use of approximate solution methods, one of which is the finite element method. The functional formulation of the problem [1] is expressed in the following form:

$$\Phi=\int_V \frac{1}{2}\left[\bar{r}K_{rr}\left(\frac{\partial T}{\partial r}\right)^2+\bar{r}K_{zz}\left(\frac{\partial T}{\partial z}\right)^2-2\bar{r}QT+2\rho c\frac{\partial T}{\partial t}T\right]dV+\int_{S_1} qTdS+\int_{S_2}\frac{h}{2}(T-T_b)^2\,dS \tag{3}$$

where V – volume; q – heat flow; h – heat exchange coefficient with the external environment; T_b – ambient temperature; S_1 - surface area to which heat flux is applied; S_2 - surface area where convective heat exchange occurs.

In ChEU, the area occupied by the object under consideration is divided into small finite elements. A triangular element is chosen as the finite element. Within each finite element, temperature approximation functions are constructed separately. The temperatures at the nodal points are chosen as the main unknowns. The temperature inside the triangular element (e) is approximated by a linear polynomial:

$$T^{(e)}(r,z,t)=\alpha_1+\alpha_2 r+\alpha_3 z \tag{4}$$

The temperature function is given by the following formula:

$$T^{(e)}=\left[N_1(r,z,t)N_2(r,z,t)N_3(r,z,t)\right]\begin{Bmatrix}T_1(t)\\T_2(t)\\T_3(t)\end{Bmatrix} \tag{5}$$

The following shape functions are applied to this finite element [6]:

$$N_1=\frac{1}{2\cdot A}\left[a_1+b_1\cdot r+c_1\cdot z\right],$$
$$N_2=\frac{1}{2\cdot A}\left[a_2+b_2\cdot r+c_2\cdot z\right], \tag{6}$$
$$N_3=\frac{1}{2\cdot A}\left[a_3+b_3\cdot r+c_3\cdot z\right],$$

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The surface area of a finite element is calculated by the following formula:

$$A = \frac{1}{2} \cdot \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix},$$

The coefficients included in the shape functions depend on the coordinates of the nodes and are listed below:

$$a_1 = r_2 \cdot z_3 - r_3 \cdot z_2, \quad a_2 = r_3 \cdot z_1 - r_1 \cdot z_3, \quad a_3 = r_1 \cdot z_2 - r_2 \cdot z_1,$$
$$b_1 = z_2 - z_3, \quad b_2 = z_3 - z_1, \quad b_3 = z_1 - z_2,$$
$$c_1 = r_3 - r_2, \quad c_2 = r_1 - r_3, \quad c_3 = r_2 - r_1.$$

$\left[B^{(e)}\right]$ - the gradient matrix can also be written:

$$\left[B^{(e)}\right] = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{7}$$

The thermal conductivity matrix of the body is as follows:

$$\left[D^{(e)}\right] = \begin{bmatrix} \overline{r} K_{rr}^{(e)} & 0 \\ 0 & \overline{r} K_{zz}^{(e)} \end{bmatrix} \tag{8}$$

where $\overline{r} = \left(r_1 + r_2 + r_3\right)/3$ represents the distance from the axis of symmetry to the center of the element.
The differential equation with respect to time for the element *e* is as follows:

$$\frac{\partial T}{\partial t} = \left[N(r,z,t)\right]_e \frac{\partial}{\partial t} \{T\}_e \tag{9}$$

For all m finite elements, we can substitute expressions (4) - (9) into expression (3) to obtain:

$$\Phi = \sum_{e=1}^{m} \left[\frac{1}{2} \cdot \int_{V^{(e)}} \left\{g^{(e)}\right\}^T \cdot \left[D^{(e)}\right] \cdot \left\{g^{(e)}\right\} dV - \int_{V^{(e)}} \left(rQ^{(e)} - \lambda \frac{\partial T}{\partial t} \right) \cdot T^{(e)} dV + \int_{S_1^{(e)}} T^{(e)} \cdot q^{(e)} dV + \right.$$
$$\left. + \int_{S_2^{(e)}} \frac{1}{2} \cdot \left[(T^{(e)} - T_b)^2 \right] \cdot h^{(e)} dS \right] \tag{10}$$

As a result of minimizing functional (10), the following system of equations is formed:

$$\frac{\partial \Phi}{\partial \{T\}} = \frac{\partial}{\partial \{T\}} \sum_{e=1}^{m} \Phi_e = \sum_{e=1}^{m} \frac{\partial \Phi_e}{\partial \{T\}} = 0 \tag{11}$$

The contribution of each finite element to the total sum (11) can be expressed as a matrix differential relation:

$$\frac{\partial \Phi_e}{\partial \{T\}} = \{Q\} = [C]_e \frac{\partial}{\partial t} \{T\}_e + [K]_e \{T\}_e - \{Q\}_e^q - \{Q\}_e^g - \{Q\}_e^h \tag{12}$$

where is the thermal conductivity matrix of the element:

$$[K]_e = \int_{V_e} [B]_e^T [D]_e [B]_e \, dV + \int_{S_{3e}} h [N]_e^T [N]_e \, dS, \tag{13}$$

The heat capacity matrix of the element:

$$[C]_e = \int_{V_e} \rho c [N]_e^T [N]_e \, dV. \tag{14}$$

The heat flux vectors at the node are the heat flux density q, the heat source Q, and the convective heat transfer coefficient, respectively:

$$\{Q\}_e^q = -\int_{S_{2e}} q [N]_e^T \, dS \tag{15}$$

$$\{Q\}_e^g = \int_{V_e} \overline{r} Q [N]_e^T \, dV \tag{16}$$

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$$\{Q\}_e^h = \int_{S_{3e}} hT_\infty [N]_e^T \, dS \tag{17}$$

Summarizing the contributions of all elements (11), a system of differential equations is formed:

$$[C]\frac{\partial}{\partial t}\{T\} + [K]\{T\} = \{Q\}^q + \{Q\}^g + \{Q\}^h \tag{18}$$

where $[K]$ - generalized heat transfer matrix; $[C]$ - generalized heat capacity matrix; $\{Q\}^q$ - heat flux vectors at the node; $\{Q\}^g$ - heat source vector at the node; $\{Q\}^h$ - convective heat transfer vector at the node.

Let us consider the solution of the differential equation (18) by the finite difference method using the central difference scheme. This equation is written in the following form:

$$[C]\frac{\partial}{\partial t}\{T\} + [K]\{T\} = \{Q\} \tag{19}$$

where $[Q] = \{Q\}^q + \{Q\}^g + \{Q\}^h$.

The derivative of the generalized vector $\{T\}$ at the midpoint of the time interval $\Delta t = t_{n+1} - t_n$ is expressed as follows.

$$\frac{\partial}{\partial t}\{T\} = \frac{1}{\Delta t}\left(\{T\}_{n+1} - \{T\}_n\right) \tag{20}$$

The generalized temperature and nodal point load vector at this midpoint of the time interval is calculated as follows:

$$\{T\} = \frac{1}{2}\left(\{T\}_{n+1} + \{T\}_n\right) \tag{21}$$

$$\{Q\} = \frac{1}{2}\left(\{Q\}_{n+1} + \{Q\}_n\right) \tag{22}$$

Substituting expressions (20) - (22) into the differential equation (19), we obtain the following recurrent formula[8]:

$$\left([K] + \frac{2}{\Delta t}[C]\right)\{T\}_{n+1} = \left(\frac{2}{\Delta t}[C] - [K]\right)\{T\}_n + 2\{Q\} \tag{23}$$

Knowing the temperature at the node at the beginning of the time interval, the temperature at the end of the time interval can be determined using formula (23). When the thermophysical properties (thermal conductivity, specific heat capacity, thermal conductivity during convection) are independent of temperature, the matrices are calculated until equation (23) is solved. If the thermophysical properties depend on temperature, then the equation is nonlinear and must be solved by iteration methods.

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III. SIMULATION&RESULTS

Problem 1: The transient heat conduction problem in an axisymmetric body made of steel is considered, where the objective is to determine the temperature distribution within the body. The cross-sectional view and dimensions of the axisymmetric body are shown in Figure 2.a. The inner surface of the body is subjected to a constant heat input of 100°C. Heat exchange occurs between the lateral surfaces of the body and the external environment, which is at a temperature of 20°C. The heat transfer coefficient between the lateral surfaces and the external environment is given as $h = 10 W/(K \cdot m^2)$. The initial temperature ($t = 0 s$) of the body is uniform at 20°C. Steel has the following thermophysical properties: $\lambda = 46 W/(m \cdot ^\circ C)$, $\rho = 7800 kg/m^3$, $c = 460 J/(kg \cdot ^\circ C)$.

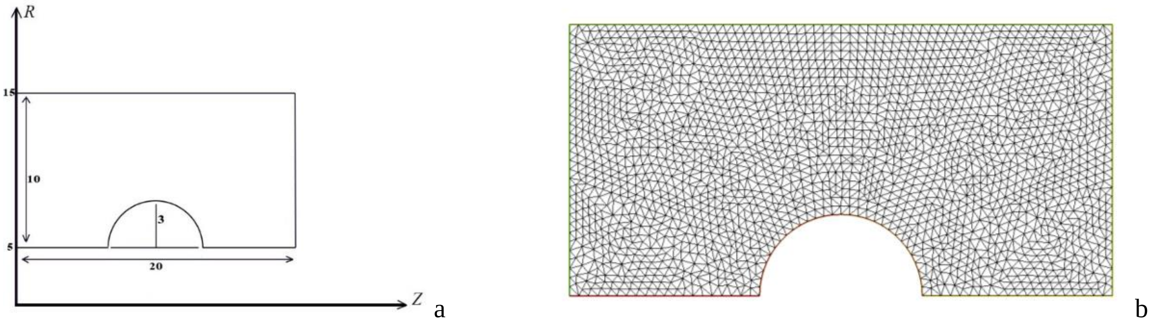

Fig 2. Axisymmetric body cross-section and finite element mesh.

To verify the reliability of the obtained results, a computational experiment was conducted to numerically study the effect of increasing the number of finite elements on the convergence of the solutions. Table 1 presents the number of finite elements and nodes in the discrete model for various configurations.

Table 1. Finite elements and number of nodes

Options	1	2	3	4
Finite elements	156	622	1388	2492
Nodes	98	350	752	1323

The temperature readings at the control points over 60 seconds are presented in Table 2 (with a time step of $\Delta t = 6 s$). The analysis of the experimental results confirms the convergence of temperature values due to the increased number of finite elements. Figure 2.b shows the fourth configuration of the finite element mesh. The numerical results of the problem solved for the 60-second temperature field using the fourth discrete model configuration, along with the visualization and isotherms, are presented in Figure 3.

Table 2. Temperature at 60 second control points (ºC)

Variantlar	koordinata (10sm, 10sm)	%	koordinata (20sm, 15sm)	%
1	75,823	4,3	50,297	6,4
2	72,554	1,9	47,062	3,5
3	71,108	0,14	45,394	0,22
4	71,005		45,494	

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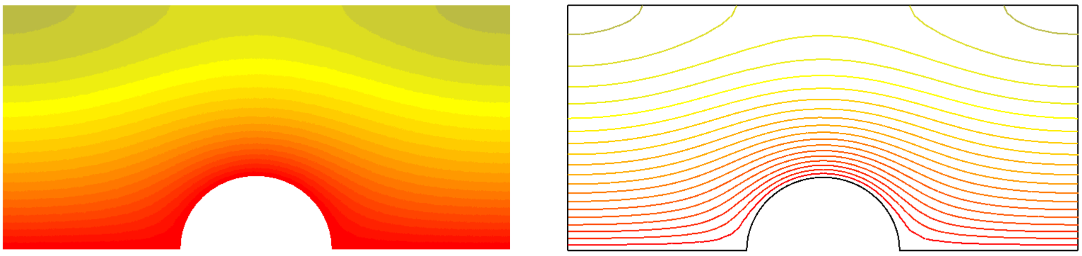

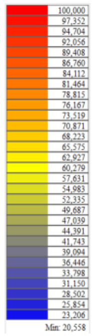

Fig 3. 60-second visualization of the temperature field and isotherms.

Problem 2: The transient heat conduction problem is considered for an axisymmetric copper body that includes two additional layers (Figure 5.a). The inner surface of the body is subjected to a constant heat input of 100°C. Heat exchange occurs between the lateral surfaces of the body and the external environment, which is at a temperature of 0°C. The heat transfer coefficient between the lateral surfaces and the external environment is denoted as $h = 10W/(K \cdot m^2)$. The initial temperature ($t = 0\,s$) of the body is uniform at 50°C. Copper possesses the following thermophysical properties:

$$\lambda_1 = 384\,W/(m \cdot {}^\circ C),\ \rho_1 = 8800\,kg/m^3, c_1 = 381\,J/(kg \cdot {}^\circ C)$$

Thermophysical parameters of the additional coating material: steel (2, in Fig. 5.a):

$$\lambda_2 = 46\,W/(m \cdot {}^\circ C),\ \rho_2 = 7800\,kg/m^3, c_2 = 460\,J/(kg \cdot {}^\circ C)$$

and iron (3, in Figure 3.a):

$$\lambda_3 = 71W/(m \cdot {}^\circ C),\ \rho_3 = 7900\,kg/m^3, c_3 = 460\,J/(kg \cdot {}^\circ C)$$

A general view of the finite element mesh of a non-homogeneous axisymmetric solid cross-section is presented in Figure 4.b.

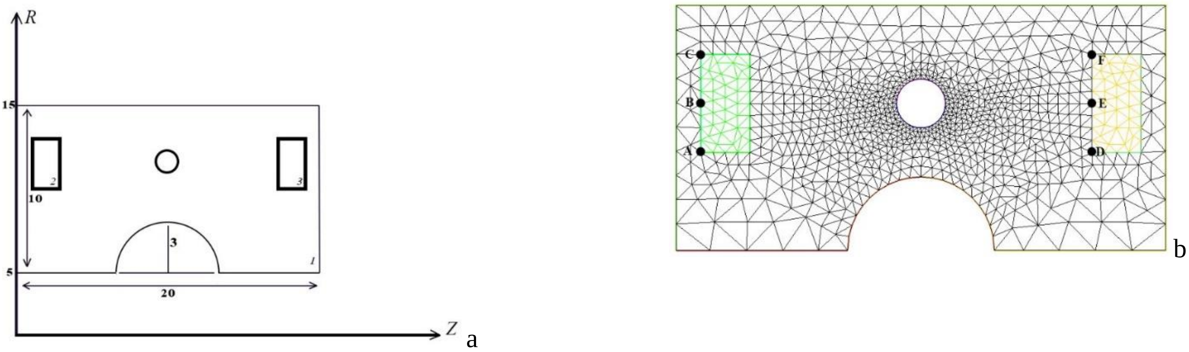

Fig 4. Cross-section and finite element mesh of an axisymmetric body.

The finite element mesh used to solve the problem includes the following parameters: 979 nodes, 1848 finite elements, a system of equations with a dimension of n=979, a bandwidth of 34 for nonzero elements, and a total simulation time of 600 seconds. Table 3 compares the numerical values at the control points in the cross-section of the axisymmetric body at t=30 seconds for homogeneous and non-homogeneous cases. The results indicate that due to the differing physical properties of iron and steel materials, the temperature values exhibit distinct variations.

Table 3. Numerical values of the temperature field

Points(cm)	A(1, 9)	B(1, 11)	C(1, 13)	D(17, 9)	E(17, 11)	F(17, 13)
homogeneous	96,554	95,433	94,719	96,697	95,548	94,791
non-homogeneous	96,478	94,418	93,295	96,698	94,957	93,751

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The graphs of temperature variations for homogeneous and non-homogeneous axisymmetric bodies at t = 5, 10, 15, 30, 45, and 60 seconds with r = 10 cm are shown in Figure 5. An analysis of the results demonstrates that the algorithm developed for solving the problem using the finite element method (FEM) accurately accounts for the geometric and physical parameters of the axisymmetric body. A comparison of the temperature distribution curves (Figure 5) shows that the resulting curves stabilize over time. To ensure accuracy, Figure 6 presents the isotherms at t = 5, 10, 15, and 30 seconds. Over time, the influence of the additional layers in the non-homogeneous axisymmetric body becomes increasingly evident.

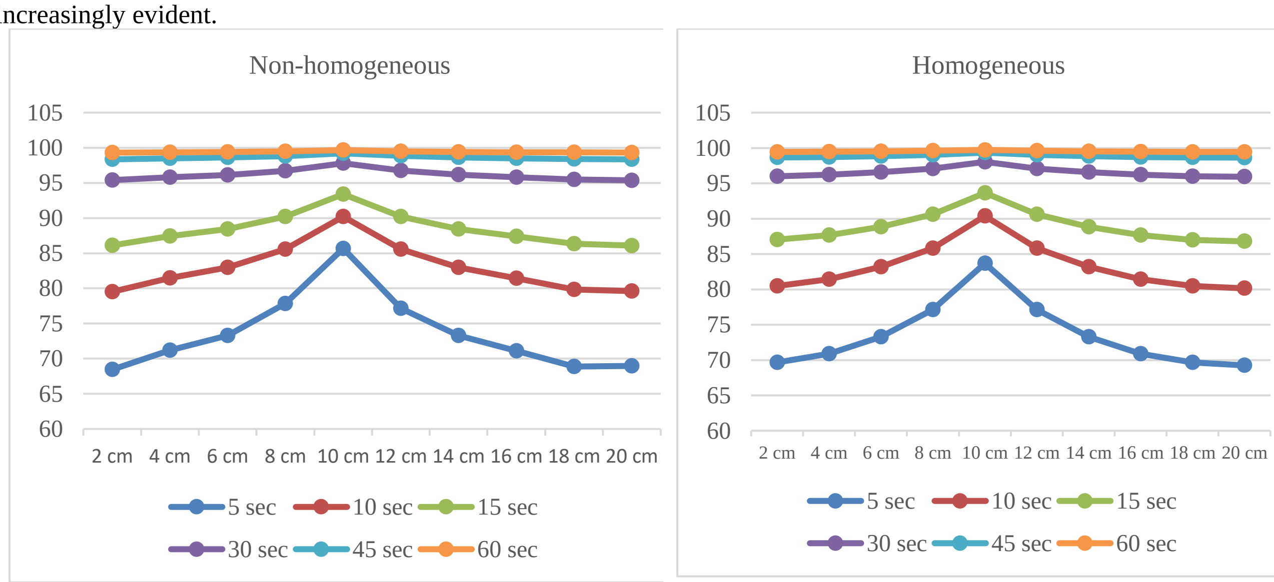
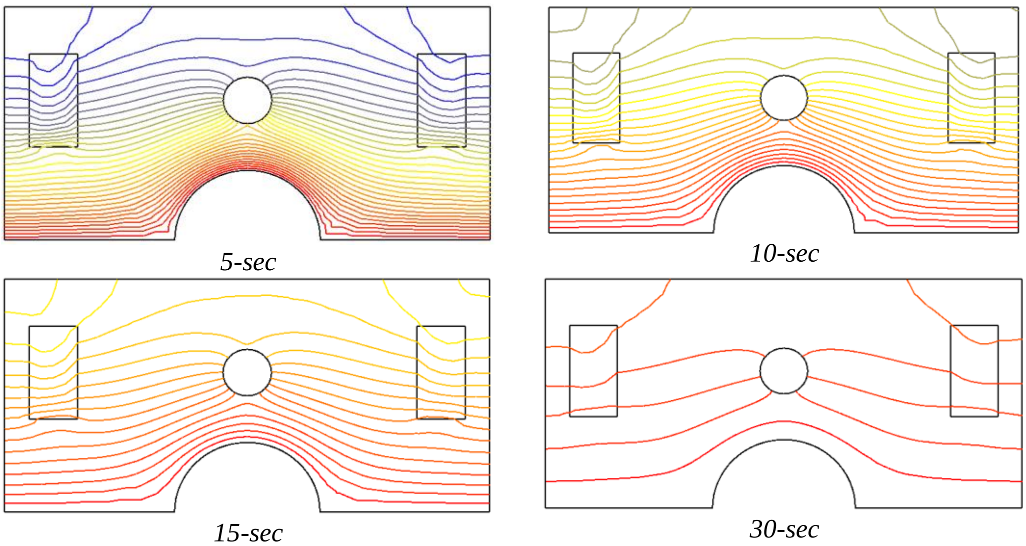


Fig 5. Change in body temperature along a horizontal distance r=10 cm.



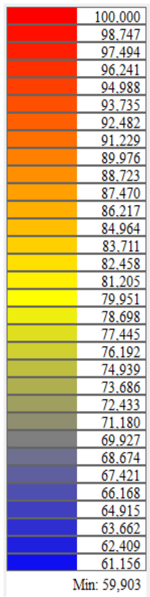


Fig 6. Isotherms at t = 5, 10, 15 and 30 seconds.

IV. CONCLUSION

To verify the accuracy of the algorithm for solving the transient heat conduction problem in an axisymmetric body, a computational experiment was conducted by increasing the number of elements. The transient heat conduction problems for both homogeneous and non-homogeneous axisymmetric bodies were solved, and the temperature values at control points were determined and analyzed. The results indicate that due to the differing physical properties of the materials and the presence of a circular cross-sectional cavity in the body, the temperature field undergoes redistribution.

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